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ABSTRACT

This proceedings from the conference of the International Group for the Psychology of Mathematics includes the following plenary addresses and invited papers: "The Social Psychology of Mathematics Education (Alan J. Bishop); "Mathematics and the Visual Arts" (Frederick van der Blij); "The In erplay between Different Settings. Tool-Object Dialectic in the Extension of Mathematical Ability - Examples from Elementary School Teaching" (Regine Douady); "The Need for Emphasizing Various Graphical Representations of 3-Dimensional Shapes and Relations" (Claude Gaulin); "Conceptual Analysis of Mathematical Ideas and Problem Solving Processes" (Richard Lesh); "Rational Analysis of Realistic Mathematics Education - The Wiskobas Program" (Adri Treffers & Fred Goffree); and "Taking Responsibility in School Mathematics Education" (Hassler Whitney). (MKR)

PROCEEDINGS OF THE NINTH INTERNATIONAL CONFERENCE FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION

Volume 2: Plenary Addresses and Invited Papers

Edited by

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INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION
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ALAN J. BISHOP

THE SOCIAL PSYCHOLOGY OF MATHEMATICS EDUCATION

I would like to begin my talk by presenting to you three items of data from different pieces of research into the learning of mathematics. I have selected these because between them they seem to be to 'catch' the essence of my title. I hope also that these items will relate to your own personal store of evidence, so that you will be able to see their significance in a wider context than that within which I shall present them.

The first concerns a secondary mathematics teacher, Alec, (MacPherson, 1973) who was deliberately trying to affect what we called the 'working relationship' between himself and certain of his pupils (is was a similar idea to that of the 'didactical contract' developed by Brousseau (1981), although it was not so directly concerned with negotiating the conditions of classwork). In this case Alec had set himself some tasks to improve the working relationship with a few of his pupils with whom he felt he had not got a good relationship, e.g. he was finding out more about their hobbies, talking to them every lesson, and only asking them a question publicly when he already felt sure that they knew the answer. In his school there were class-orders kept in each subject, and these were reviewed each half-term, so that he was able to see if his behaviour had any effect on the pupils' order in class. With all pupils his behaviour had important effects and with one girl a surprising fact was revealed – not only did *she* change here class-order from 20th place to the 4th place in mathematics, but also her best friend, who sat next to her, moved up as well.

In some way Alec's influence was communicated to another pupil than just the one he was trying to influence. I was reminded of Jacob Kounin's (1970) study of discipline and group management, where he explored what he called the 'ripple effect' of various 'desist techniques'.

The second piece of reported data is from Lorenz' (1982) study where he described particularly the ways in which mathematics teachers thought about their pupils and how these views manifested themselves in the classroom. Amongst various fascinating findings was one which stood out for me, because it was a paradigmatic example of a phenomenon I had seen in many classrooms of both beginning and experienced teachers. He found that teachers' behaviours which were designed to be 'helpful' were in fact directed more often towards the more-able pupils than to their less-able, but presumably more needy, peers.

The third item of phenomena is one which I have reported before (Bishop, 1979). I make no apology for reporting it again because it continues to fascinate me. The fact that I was directly involved is also important because it illustrates the point that it has for me a social, and indeed an emotional significance, as well as a cognitive significance.

It concerns me interviewing a university student in Papua New Guinea and trying to understand more about his 'local' or 'folk' mathematics. I asked him how he would find the area of a rectangular piece of paper. He replied:

"Multiply the length by the width". "You have gardens in your village. How do your people judge the area of their gardens?" "By *adding* the length and the width". "Is that difficult to understand?" "No, at home I add, at school I multiply". "But they both refer to area". "Yes, but one is about the area of a piece of paper and the other is about a garden". So I drew two (rectangular) gardens on the paper, one bigger than the other. "If these were two gardens which would you rather have?" "It depends on many things, I cannot say. The soil, the shade . . ." I was then about to ask the next question "Yes, but if they had the *same* soil, shade . . ." when I realised how silly that would sound in that context.

Now it would be relatively easy to dismiss the first item as some sort of accidental coincidence. The second item would be harder to dismiss but could be explained by criticising the accuracy of teachers' knowledge of who were the more- and less-able pupils in their classes. The third item seems set up for a perfect piece of resolution by the teacher of a learner's cognitive conflict.

However I do *not* want to dismiss them, nor do I want to try to find essentially cognitive explanations for them. They interest me because they are typical of many situations which have a strong *social* component, and which I feel have been relatively ignored by researchers. In the context of this talk they represent phenomena and problems in the area of the social psychology of mathematics education. I hope also, as I said earlier, that these three items have some personal meanings for you as well because whilst describing my ideas about social psychology I am also trying to influence you. I can only do that if you are socially involved with these problems as well as being intellectually motivated to attend to them.

A learning experience like this, and I hope this is a learning experience for you, is as much a social experience as it is a cognitive one. For example, learning from other people *is* different from learning from texts and a context such as this does have a social dimension to it. I know that listening to me talking today *is* a different experience from reading the printed text. I am not suggesting or implying that one is a *better* ex-

perience than the other but I hope we can agree that they are certainly different experiences. Therefore, *if* the aim of research into the learning and teaching of mathematics is to understand more how these happen, we must attend to this social dimension, since mathematics learning in classrooms, by definition, takes place in a social context. Mathematics classrooms are very 'public' places in which it is impossible to achieve privacy. Every act is performed in a social situation even if it involves pupils using their own individual text materials. Every interaction between a pupil and some mathematics in the classroom is socially mediated. As with the classic research of Asch (1951) even if an individual pupil believes a certain mathematical proposition to be true, the social and interpersonal influences operating in the classroom can prevent the pupil expressing that proposition 'publicly', and can also make the pupil think she is wrong.

Fortunately research in this social area is growing and it is not as deserted a terrain as once it was. We have seen developments in research on topics like the fear of mathematics, sex-role stereotyping, pupils' attitudes and attributions, teachers' perceptions and epistemologies, and collaborative learning, all of which can increase our awareness and understanding of social phenomena in the learning of mathematics. What I should like to do today is to help increase the momentum of this research, to help coordinate and connect some of the developments and to help identify the significant aspects from the perspective of teacher training and teacher education.

Firstly I think it is necessary for today's talk to set the "social psychology" emphasis in context of my general views on the social dimension in mathematics education. Research into this dimension is significant, for me, at five levels. At the *cultural* level research can inform us about the history and development of mathematical ideas and their relationship with one's culture (e.g. Kline, 1954). Also cross cultural studies like Lancy's (1983) and analyses like Ellul's (1980) and Weizenbaum's (1976) sensitise us to more complex aspects of this relationship.

At the *societal* level, the research concerns the various institutions in society and the political and ideological influences which they bring to bear on the mathematics education of our children (see for example Griffiths and Howson, 1974; Swetz, 1978). Some of these institutions are formally concerned with education of course but many are not and accounts like Fasheh's (1982) illustrate well the tensions and conflicts which exist between them.

At the *institutional* level research is about for example the within-school influences which help to shape the intended and the implemented mathematics curriculum for the pupils (see for example, Stake and Easley, 1978). Donovan's (1983) study also concerns these influences and shows how the values and ideologies of the dominant cultural group filter into the institution of school. Marrett and Gates (1983) describe how such values determine which pupils study mathematics in which tracks (or sets) and thereby indicates another institutional mechanism for controlling the pupils' mathematical education.

At the *pedagogical* level we at last enter the classroom and find research some of which relates specifically to our topic today. I have added another level to the social dimension though which I call the *individual* level, because there is a growing amount of research which focusses on the learner from a social perspective. This again I shall say more about.

I hope this brief overview serves to demonstrate that the social and interpersonal influences on the learner in the mathematics classroom have strong connections with values and ideologies emerging from interactions taking place far from the classroom. An awareness of the whole social dimension reminds us I hope of these connections. If there is one thing to be learned from research into social aspects of mathematics education it is that the *context*, and the *situation*, are all important.

Concentrating now on social psychology, I want to look at three aspects today: social motivation, social cognition and social interaction.

1. SOCIAL MOTIVATION

Let us begin with a topic which has stimulated a great deal of research activity world-wide – namely, the fear of mathematics. It is a topic which has been fruitfully analysed by Buxton (1981) amongst others and which contains many ideas of importance to teachers and teacher educators. Of particular interest here is the fact that both Buxton and Skemp (1979) use the idea of goals and anti-goals (to be avoided by the learner) and their discussions of anxiety, frustration and other emotions are very helpful to our understanding of how the classroom mathematical experience appears to pupils.

Another anti-goal identified in the literature is the 'fear of success' construct found to be of great value by Leder (1980) in understanding why bright girls in mathematics deliberately avoid success and achievement in order to retain the respect and acceptance of their peers. This is

of course not just a phenomenon to be seen with bright girls. It will be noticed by any teacher of mathematics particularly of adolescent children, who will apparently prefer not to succeed and indeed, not to try to succeed for fear of losing the respect of their friends. At the adolescent stage, well noted for being a time of questioning and challenging authority, goals promoted by the teacher may well be perceived as anti-goals by some pupils (hopefully not all!).

Whether the teacher-mediated goals are accepted by the pupils *as* goals, or converted into anti-goals, will be determined by various factors. In particular the role of Significant Others must be recognised. Although this idea (S.O.) was developed by Sullivan (1940) within the psychiatric field it does have value for us also. You do not require much observation time in mathematics classrooms to begin to identify which individuals in the group significantly affect the behaviour and the motivations of others. The situation presented at the start of the talk illustrates this. The pupil whom Alec was trying to influence was also clearly a Significant Other for her neighbour and the change in motivation and achievement in one had a very strong affect on the other pupil.

Of course it is likely that for many pupils the teacher will have the status of a Significant Other. But it is also true that for some pupils this will not be the case. Likewise there will often be some pupils who will become S.O. for the teacher, and will have a significant shaping effect on the teacher's motivation and behaviour. This point reminds us that teachers can also have goals and anti-goals, with e.g. the "development of mathematical understanding" being a clear goal and the "fear of confrontation" being a strong anti-goal for many teachers. Again we can understand how individual pupils, acting as Significant Others for the teacher, can affect the relative strength of the teacher's goal/anti-goal tension.

In an earlier paper (Bishop, 1981), I presented some ideas concerning *mathematical involvement*, a construct designed to describe affect-in-action, i.e. the observable realisation of a positive attitude towards mathematics. It concerns the extent to which pupils demonstrate their willingness to engage in a class' mathematical activity. Given the goal of the teacher to try to increase the mathematical involvement of as many pupils as possible, it would also be an indicator of the extent to which any one pupil related to the teacher as Significant Other. Furthermore the teacher as a leader, or as a model, would also clearly be an important factor in creating mathematical involvement.

Finally in the section I should like to mention exchange theory – a motivation theory which essentially proposes that individuals engage in interactions which offer them more as *rewards* than they are giving out as *costs* (Homans, 1961). In these terms it is unrealistic of the teacher to imagine that 'motivation' is a once and for all problem, e.g. that once the child is motivated to do well at mathematics that motivation will carry on through the year and through the school. Equally it offers an alternative view of motivation from that of the verb 'to motivate'. It implies instead that teachers recognise that pupils will only become involved in a mathematical activity if the perceived 'rewards' are greater that the perceived costs (potential loss of friendships, mental strain, fear of failure, etc.). Furthermore it predicts that once the costs exceed the rewards, the involvement will cease. Despite this theory's simple and perhaps 'too-mercenary' view of human nature, it does nevertheless help to explain and predict many of the paradigmatic problems of social motivation.

Perhaps we need more research which looks at pupils' goals (and anti-goals) in relation to Significant Others? Perhaps we need to resuscitate the old ideas of sociometrics and sociograms, but instead of looking merely for friendship groupings and for isolates, we should look more for the S.O.'s who influence choices of goals or anti-goals? Finally, perceptions of the 'rewards' and 'costs' of mathematical involvement by different pupils would also be of importance together with the relevant perceptions of their S.O.'s, one of whom may be the teacher. Once again this kind of analysis shows us that research which considers only the teacher as *the* influence on the pupil will probably miss the real influences.

2. SOCIAL COGNITION

This section concerns the ways in which people 'know' other people, and in relation to mathematics classrooms we are particularly interested in the ways the pupils are known. Teachers' perceptions about their pupils has been a fertile ground for research for many years and their importance was well demonstrated by studies of their 'self-fulfilling prophecy' – whereby pupil's live up to, or live down to, the perceptions and expectations of them by their teachers. It seems to me moreover that what is pedagogically significant about *any* psychological pupil construct is how that phenomenon is perceived by the teacher. Even if a researcher 'establishes' that, for example, a pupil has a preference for using visual

imagery in mathematics, what really matters is the teacher's perception of that situation. As another example, I found it interesting to analyse teachers' responses to pupils' errors by using the idea of teachers' *perceived error* (Bishop, 1976).

Mention of the word 'construct' above requires that I give due recognition to Kelly's (1955) Personal Construct Theory, a theory which many researchers now use implicitly to guide their work. At the heart of the theory lies our individual system of bi-polar constructs and one construct which, in my experience, many mathematics teachers use, both implicitly and explicitly to shape their constructions of their pupils, is that of 'mathematical ability'. Teachers' behaviours seem to be strongly affected by their perceptions of the more-able/less-able 'dimension' and I would like us to be clear that when we are discussing aspects of teaching like this with teachers, we call it *perceived* mathematical ability. Labels like 'mathematical ability' have a way of becoming very fixed and stable classifications in many teacher's minds, and they need reminding that they are only talking about 'perceptions' which one can, and should, be prepared to change.

If we link this idea with another we can see some important consequences. Various researchers have considered the particular problems faced by girls in learning mathematics, and amongst other ideas which have been explored is that of sex-role stereotyping. This label is put on the behaviours of teachers and others which seem to restrain girls' behaviours so that they stay close to a certain role-model for girls. Researchers like Becker (1981) have identified the obvious and not so obvious ways in which teachers do this.

If however we consider the general idea of role-stereotype, we can see how other groups of learners are made to become disadvantaged in the same way as some girls are. For example, there are undoubtedly many instances of ability-role stereotyping, whereby teachers' behaviours towards more-able pupils are markedly different from their behaviours towards less-able pupils. One would naively assume that these different behaviours are designed to improve the performance of the less-able pupils, but this (as the data from Lorenz' study shows) is *not* what happens. The way to understand this phenomenon is to treat it as role-stereotyping, whereby the more-able pupils are encouraged to be more-able and the less-able pupils are encouraged to continue to be less-able.

One has of course also seen many instances of class-role stereotyping (upper, middle, and working classes) and of race-role stereotyping but I have come across another situation which also surprised me. I call it

handicap-role stereotyping which I have seen with both blind and deaf children who are kept playing a dependent and 'appropriate' handicapped role. It is, furthermore, very difficult for any mathematics teacher who wants to break away from the stultifying effects of any of this stereotyping if the educational system continues to support it. In the U.K., for example, 'setting' the pupils into so-called homogeneous ability groups for mathematics occurs in almost all secondary schools. Such an institutionalised system clearly reinforces the ability-role stereotyping which many teachers adopt. In my personal view this is a far more serious and widespread problem than sex-role stereotyping nowadays.

One way to get beyond mere stereotyping is perhaps to make teachers more aware about how their behaviours and expections shape pupils' *attributions*. The interest in attribution theory has grown in recent years and there is a well-developed literature (see Weiner, 1972, for example). One strand of the research looks at children's perceived causes of their performance and whether these causes are internal or external to the pupil. Another strand considers teachers' attributions of pupils' performance. For example, Johnson *et al.* (1964) taught pairs of 10 year old children some arithmetic procedures. For each pair it was organised that one child (*A*) would do well at the first assessment while the other (*B*) would do poorly. The teachers then taught each pair again and this time, while *A* continued to do well, it was arranged that half of the *B* pupils improved and the other half declined. Amongst other findings was the interesting one that the teachers attributed the improving *B*'s performance to their teaching, but they attributed the declining *B*'s performance to the pupils themselves.

Clearly attribution theory could help teachers to understand more about their role in pupils' development of their own self-concept. What would be important to know more about is how, and under what conditions, attributions can change. Once again Alec's story, from earlier, gives us some indications, but if all of us are not to remain trapped by our own attributions we must try to interpret this idea much more dynamically. Kelly discusses a treatment he calls 'fixed-role therapy' which I applied to the idea of teachers doing more of their own research and investigations (see Bishop, 1972). That was the context from Alec's story arose. It was clear to me, and to him, that the 'therapy' of playing a 'fixed-role' (the researcher) for a period of time, had the effect of changing dramatically his perceptions, his constructions and his attributions. It would be useful to have more evidence of such changes.

And why not make an imaginative analogue here? If *that* strategy helped to change a teacher's attributions, could a similar strategy affect

a pupil's attributions? But what could such a strategy look like?

3. SOCIAL INTERACTION

We now turn to research and ideas which focus more explicitly on the processes of social interaction. All the time we have been discussing motivation and cognition the social interaction processes have had an implied presence and effect, but now we should consider some aspects a little more directly.

First of all we find that the literature sensitizes us to the distinction between 'communication' and 'influence'. I feel also that it is important to distinguish between these because the relative position of the interactors implied by the two is different, and has therefore different consequences for the teacher. For example, many teachers having asked a pupil a question, then only prepare to evaluate and judge the answer received. Indeed the position of 'evaluator' also predisposes teachers to ask certain kinds of questions rather than others. The difference between communication and influence is illustrated well by this extract from Harvey *et al.* (1982) in their study of language in mathematics:

D. 15's odd and a ½'s even.
RH. 15's odd and a ½'s even? Is it?
D. Yes.
RH. Why is a ½ even?
D. Because erm, ¼'s odd and ½ must be even.
RH. Why is ¼ odd?
D. Because it's only 3.
RH. What's only 3?
D. A ¼.
RH. A ¼'s only 3?
D. That's what I did in my division.

At this point another child joined in to explain to the teacher:

R. Yes, there's three parts in a quarter like on a clock. It goes 5, 10, 15.
RH. Oh, I see.
R. There's only three parts in it.
RH. Ah, so you've got three lots of 5 minutes makes a quarter of an hour.
D. Yes. No. Yes, yes, yes.

(Harvey *et al.*, 1982, p. 28)

The teacher *could* have evaluated the pupil's response at various points, which would then probably have led to attempts to change the pupil's view – that is the shift from communication to influence. There is a danger, I feel, in the teacher only using the influential mode rather than appropriately (as in the case above) using the communication mode. I have chosen this example because, in my experience, such examples of communication in mathematics classrooms are rare. They need to be better documented in research, I feel.

But communication is not only verbal, as many studies and our own experience tell us. It is also a two-way process in that both transmitter and receiver must play a part if communication is to occur. Moreover it is not always intentional on the part of the transmitter, and unintentional 'messages' are conveyed around the classroom by gesture, bodily position, facial reactions and by words. Such messages are the 'fabric' of social interaction from which we weave our constructions of others, and our views of ourselves as others see us. Unintentional messages convey to pupils the teacher's perceptions of them as much as do the intentional messages, and some would say more so.

It is sometimes a matter of what is *not* communicated, as Webb's (1982) research shows. She studied heterogenous small groups working on mathematical problems and found that one frequent occurrence was that questions to the rest of the group by the less-able pupils were more often ignored than accepted. One can easily predict the meanings conveyed by *that* message.

Moving now to the area of social and interpersonal *influence*, the report of Perret – Clermont and her colleagues (1984) must be quoted here. It provides us with an excellent summary of research and ideas concerning the role of other people in children's intellectual development. As well as containing many ideas of interest and value to teachers it should also provide a warning to any interviewer of any child *not* to ignore the social relationship between them. (My story from Papua New Guinea at the start of the talk should not just be interpreted from a cognitive standpoint either. The interactors had markedly different values and sets of cultural assumptions which clearly affected the interview.) Time and again we learn of children assessing the *social* situation and context of the interview in their interpretation of the task, in their responses and in their judgement of the way their responses are received. Clearly, too, in a mathematics classroom, a mathematical activity is as much a *social* activity as it is an intellectual one, and this awareness is critical for the teacher in interpreting problems of both motivation and cognition.

Whereas Piagetian interviewers may not be intentionally influencing the child, the teacher usually is, and correctly so in my opinion. But the power given to the teacher by society, and usually achieved also in the classroom, does not necessarily dictate the *kind* of influence exerted by the teacher. My own analysis based on Barnes' ideas (1976) and others is that, in terms of the pupils' mathematical development we can see an 'influence' dimension varying from 'imposition' to 'negotiation'. An imposition interaction pattern is characterised by the teacher maintaining tight controls over rules of procedure, over the kinds of acceptable contributions (unusually of a very limited nature) over the amount of talk (teacher maximum), over meanings of terms and over the methods of solution. The mathematics teacher together with the textbook would represent the mathematical authority for the validity of solutions and the transmission of ideas and meaning from teacher to pupils would be emphasised.

In a negation interaction pattern on the other hand, the rules of procedure are discussed and agreed on rather than imposed, the kinds of contributions from pupils will vary, there will be more equal amounts of teacher and pupil talk, and there will be discussions over meanings and over methods of solution. The mathematical context itself will offer the criteria for judging the acceptability and validity of solutions wherever possible, and in other cases the nature of the conventional criteria will be made explicit. In comparison with the transmission of ideas in the imposition pattern, here we would expect to find more of an emphasis on communication of ideas between teacher and pupils, and on establishing and developing *shared* meanings.

Once again it would be useful to have ideas from research about changes in interaction patterns and to know what conditions surrounded a change from imposition to negotiation, or vice versa. However, from the unintentional messages contained in my descriptions above, I am sure that you will correctly infer that my preference would be for more negotiation and less imposition!

There is much more one could say about social interaction but space only permits this brief reflection on what I feel are the most promising research developments for us in mathematics education.

POSTSCRIPT

Here then are some preliminary thoughts about what I think of as the social psychology of mathematics education. I hope my reasons for my

interest have also been communicated in the talk and I hope that I have persuaded others of the validity of these reasons.

I would certainly like to see this organisation take a lead in developing this area of research and I look forward to seeing more papers on this area presented in subsequent conferences. I would be pleased to convene a meeting during this conference to try to involve more people in the development of this research.

University of Cambridge, Department of Education

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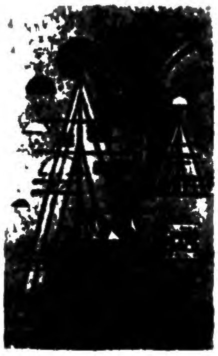
Mathematics and the Visual Arts

Frederick van der Blij

Mathematics is the majestic structure conceived by man to grant him the comprehension of the universe.

LE CORBUSIER



He said to me one day in the second week of July, "Asher Lev, there are two ways of painting the world. In the whole history of art, there are only these two ways. One is the way of Greece and Africa, which sees the world as a geometric design. The other is the way of Persia and India and China, which sees the world as a flower. Ingres, Cézanne, Picasso paint the world as geometry. Van Gogh, Renoir, Kandinsky, Chagall paint the world as a flower. I am a geometrician. I sculpt cylinders, cubes, triangles, and cones. The world is structure, and structure to me is geometry. I sculpt geometry. I see the world as hard-edged, filled with lines and angles. And I see it as wild and raging and hideous, and only occasionally beautiful. The world fills me with disgust more often than it fills me with joy. Are you listening to me, Asher Lev? The world is a terrible place. I do not sculpt and paint to make the world sacred. I sculpt and paint to give permanence to my feelings about how terrible this world truly is.

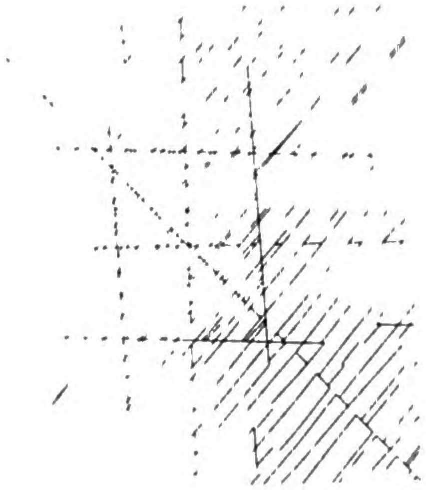
CHAIM POTOK
(My name is Asher Lev)

Kunst gibt nicht das Sichtbare wieder, sondern macht sichtbar.

PAUL KLEE

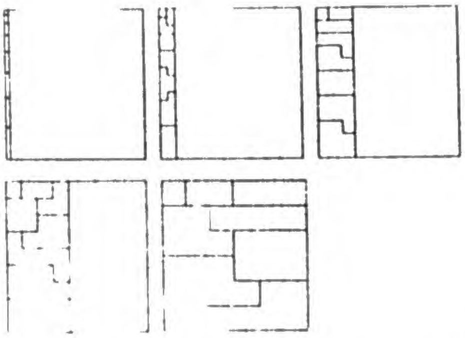
– C'est un cheval, n'est-ce pas?
– Oui, oui.
– Mais non, c'est un oiseau.
– Oui, oui.
Ainsi répond Miró, toujours poli, quand on lui demande une explication.

PIERRE LOEB



A substantial number of artists are now working in such a way that the syntactic content of their work can be exhaustively notated. This method makes possible the establishment of a coherent corpus of work with clear internal and external relationships. In his 1949 essay "Die mathematische Denkweise in der Kunst unserer Zeit" Max Bill provided a definitive refutation of the criticisms that works belonging to this category are, on the one hand, mere illustrations of the mathematical principles or, on the other hand, that their mathematical content is a mere pretext for formalist aesthetics.

JEFFREY STEELE



Comic strip is a form of serial drawing in which incidents are separated by a frame or a space. The means is simple yet it can suggest time passing.

Although my drawings have no funny people in them, they are related to the comic strip by their means of presentation. They are also related to the Bayeux Tapestry, Trajan's Column and much of Hogarth's work. They all present events in chronological order where something is finally resolved; a battle is won, a marriage ends in disaster, etc. My drawings however constitute a logical order. The marks in my serial do not refer to any set of events actual or imagined. Instead the spectator is asked to look at them for what they are.

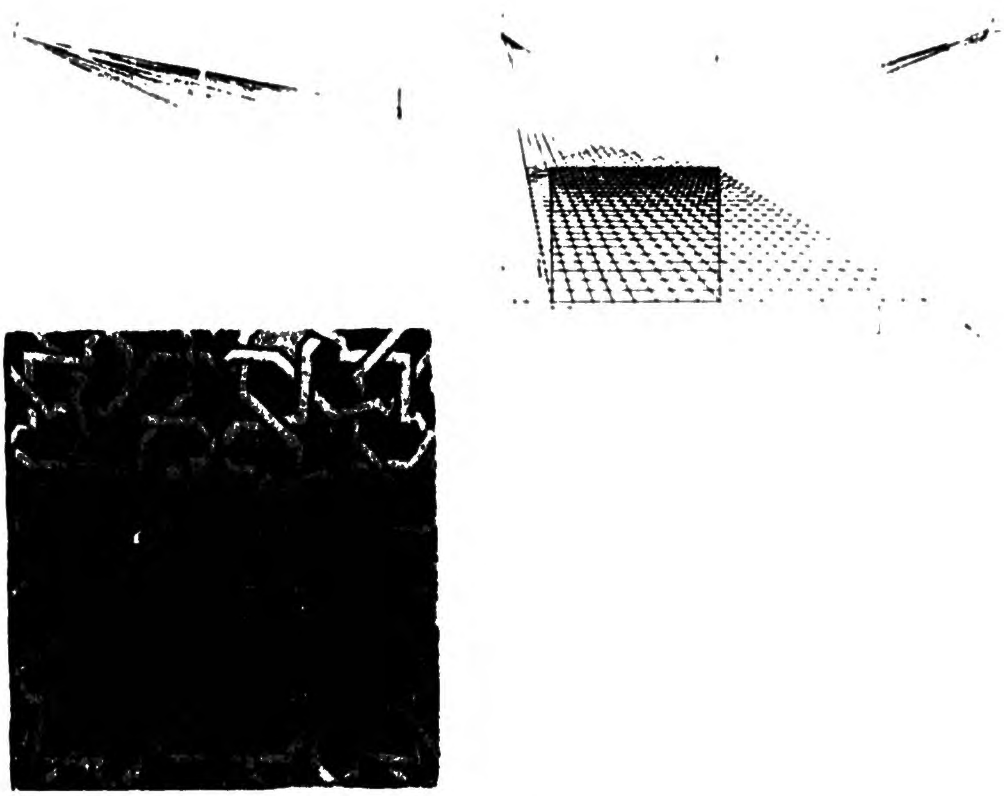
Because abstract art is free of narrative it may draw our attention to structural ideas which are often obscured in figurative art by the power of its images. My drawings are a series within a series, an eight part one within a five part one. The linear square around each drawing both separates events and is part of them. It compresses a scale of intervals which progressively expands. The scale of intervals is based on $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8$.

Because these add up to 36, it fits the following areas exactly: 1×36, 2×18, 3×12, 4×9, 6×6. In the first drawing the scale is simply placed against an edge of the square. In subsequent drawings it expands and folds against itself till the final drawing where it completely fills the square.

(1976)
PETER LOWE

MATHEMATICS AND THE VISUAL ARTS

In this lecture I'll try to investigate some relations between two features of our culture, mathematics and the visual art. I'll restrict myself, with a few exceptions, to the art of the last century. Much could be said and is already said of the rôle of mathematics in the visual arts in former times. One may think of the theory of symmetry and ornaments[1], the theory of perspective[2] and so on.



There are many, many ways to deal with our subject. I give a list of some viewpoints and I make a personal choice on which of these details I give more or less emphasis.

1. *Cubism*

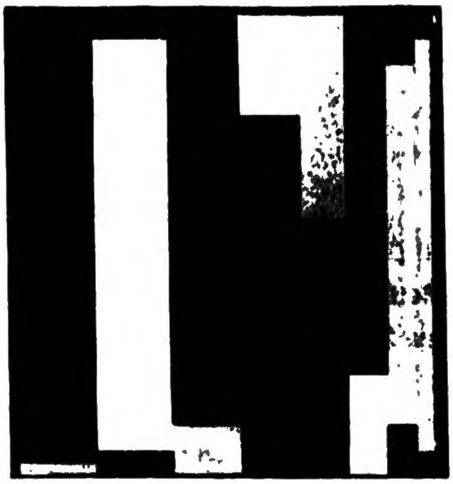
Long before the birth of cubism as an -ism, the use of single geometric approximations of still-live objects was well-known in the education of coming artists. But in cubism there is a theory to see the world as a set of geometrical forms, as it was remarked by Chaim Potok. There is no

need to recall the paintings of Cezanne, Braque, and Picasso of their cubistic period, they are well-known.

Although the use of geometrical forms, cubes, cones, and so on give some of the cubist paintings a geometrical aspect, some of the paintings of Leger and Malevich give us more an industrial or technical impression.

2. *Golden section and special ratio's*

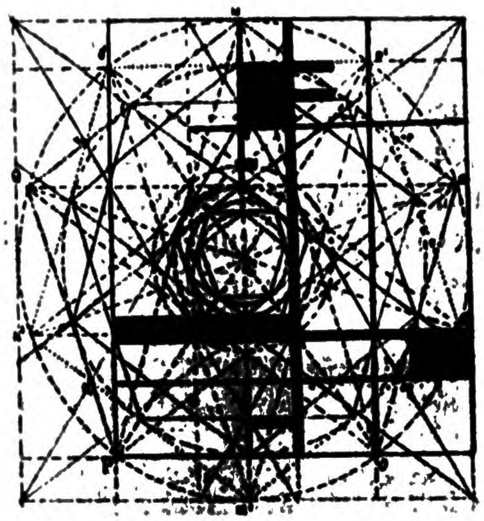
The use of special simple ratio's, such as 1 : 2, 2 : 3, known from the theory of Larmony, in drawings, paintings and architecture has a long tradition. The painter Richard Paul Lohse uses these ratio's in many ways, e.g. in "Progressiv gestufte Gruppen mit gleicher Farbzahl". The theory of Vitruvius is a classical example.



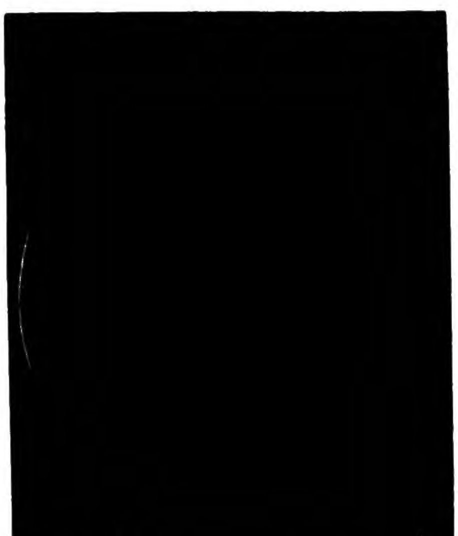
But also irrational ratio's play an important rôle in aesthetic theory. Ratio's derived from the square and the regular triangle such as $1 : \sqrt{2}$ and $1 : \sqrt{3}$ are examples. But the golden section, the ratio $1 : (-\frac{1}{2} + \frac{1}{2}\sqrt{5})$, derived from the regular pentagon, and also from the dodecaëder has been object of many books and papers.[3] To make an arbitrary choice, I found a catalogue of the Belgian artist Amédée Cortier in which he writes: "Je veux exprimer dans mes peintures un équilibre de surfaces colorées qui doit donner aux spectateurs le sentiment de bien-être". And Saskia Bos writes in this catalogue a note: "Recherches sur l'usage de la section d'or dans l'oeuvre d'Amédée Cortier". But there are many other examples to be given. In the time of the "Stijl" there was a group of artists working under the name "La Section d'Or" and there was a famous exhibition with the same title. It looks like that the most famous painter of the "Stijl" Piet Mondriaan did not use the golden section at all, however.

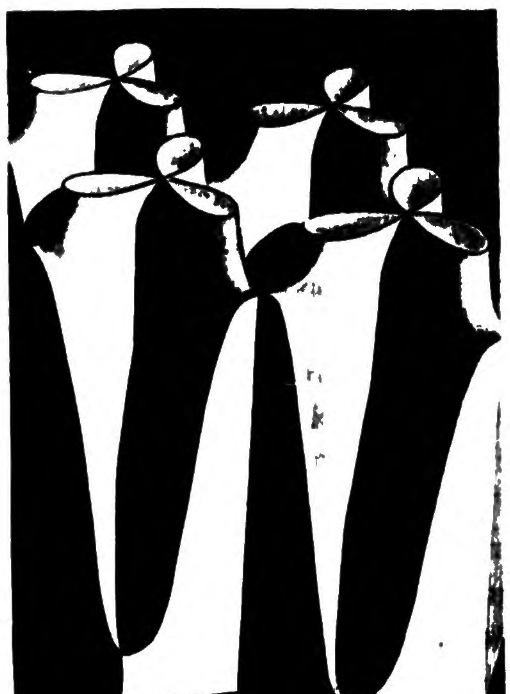
3. *Mathematical objects as a source of inspiration for the artist*

Still thinking about the golden section and the pentagon, I give a reproduction of a painting of Jean Gorin entitled: "Étude d'une com-



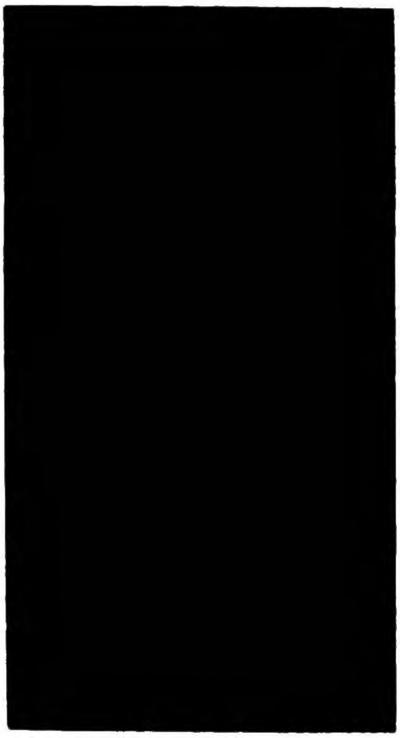
position émanant du pentagone inscrit et circonscrit d'un cercle". Max Bill made a lot of sculptures using simple geometrical forms, such as demi-spheres, but his "Endless Ribbon" is just a Möbius strip.





In quite another way the painter (and famous photographer) Man Ray uses for some of his pictures that bear names of plays of Shakespeare, objects to be found in the cabinets of old mathematical institutes, used in the beginning of this century by courses in complex functiontheory. These models of plaster, made in the style of the illustrations in the

Jahnke-Emde tables, deal with elliptic functions and can also be found in several science-musea. Man Ray used one of these models in his painting: "The merry wives of Windsor". And still in another way G. de

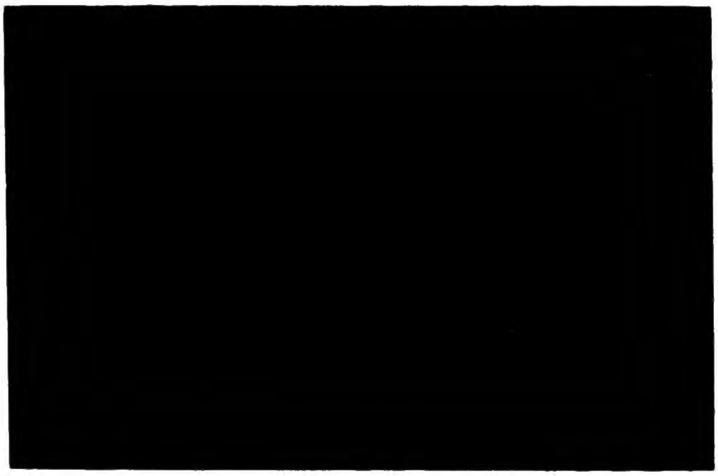


Chirico used mathematical objects in his metaphysical painting: "Il Trovatore".

We can compare this work with a portrait of Euclid by the surrealistic painter Max Ernst. In this portrait the relation between the title and the subject of the painting is obvious. But I cannot understand the meaning of the titles of some paintings of G. Vantongerloo, such as "$Y = ax^2 + bx + 18$".

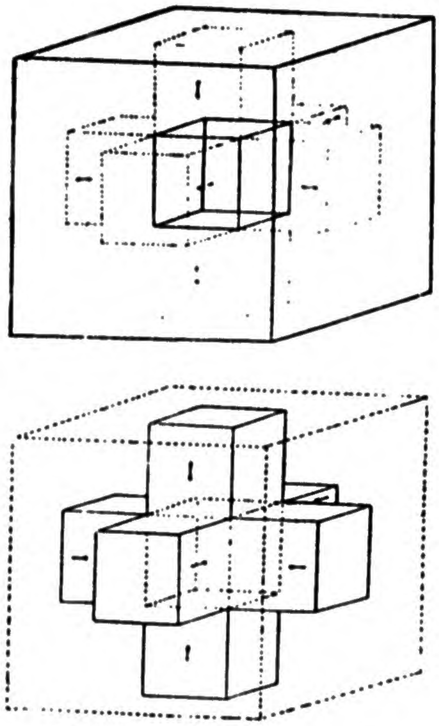
4. *The fourth dimension*

At the end of the 19th century and the beginning of the 20th century there was quite a commotion concerning the mathematical construction of geometrical objects in the fourth dimension. These were studies about regular polytopes, more special about the generalisation of the square and the cube, which is called hypercube or tesseract. A broader public became interested in the fourth dimension. Partially by popular books, like Flatland, on the subject, but also since the notion was used in theories to explain "Creatio ex nihilo" in spiritism and so on (Zöllner). Afterwards the use of the fourth dimension in the physical theory of relativity, brought in the time as the fourth dimension. And although the painter G. Severini warned his brothers in the "Stijl" that there is still a $\sqrt{-1}$ connected with the time as fourth dimension, many artists tried



to bring in time as fourth dimension in their painting. We recall that M. Duchamp in "Nude descending a staircase" used in 1912 already cinematographic theories in order to express "time" in a painting.

Quite a lot of pictures and sculptures of the fourth dimension were constructed. We give some examples. First a "Construction Spatiale aux 3^e et 4^e dimension" of Antoine Pevsner; further a drawing of a cube and a hypercube by Th. van Doesburg, one of the members of the "Stijl",



and the use of such a picture of a hypercube by Salvador Dali in his painting: "Corpus Hypercubicius". Quite another representation of the fourth dimension can be found in a picture of Max Weber: "Interior of the fourth Dimension".

There is a very complete study of the use of the fourth dimension in art by Linda Dalrymple Henderson in her book: "The Fourth Dimension and Non-Euclidean Geometry in Modern Art" (Princeton, 1983). In this study of more than 400 pages there are hundreds and hundreds of items in the bibliography on this subject.

5. *The use of abstraction*

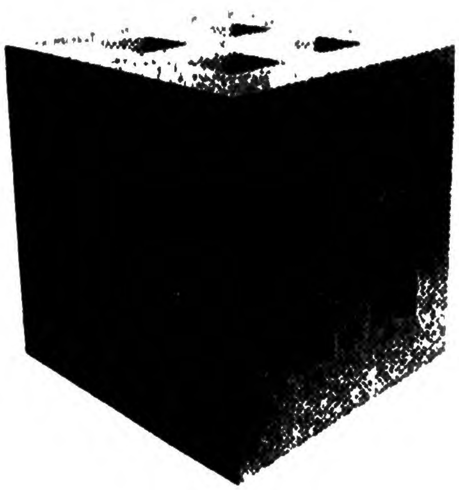
Is it a sheer coïncidence that the notions abstract mathematics and abstract art became popular in the beginning of this century? There was always abstract music, but the new dodecaphony, the twelve tone compositions, had its roots in the same time as the beginning of abstract painting by Kandinsky (there is a correspondence between Kandinsky and Schönberg). Pierre Boulez, a well known composer and conductor of our time, writes about Kandinsky: "For me, the emancipation of tonality was equivalent to the emancipation of the object – or the subject. These two adventures stimulated a considerable growth of thought and creation in their respective fields". And I propose that we can use the notion abstraction for this emancipation, and that we can describe abstract mathematics with the same motions. Potok states that Kandinsky saw the world as a flower but Kandinsky wrote a book with the title: "Point and Line to Plane" and another "Concerning the Spiritual in Art" in which we find rather abstract theories in stead of flowers.

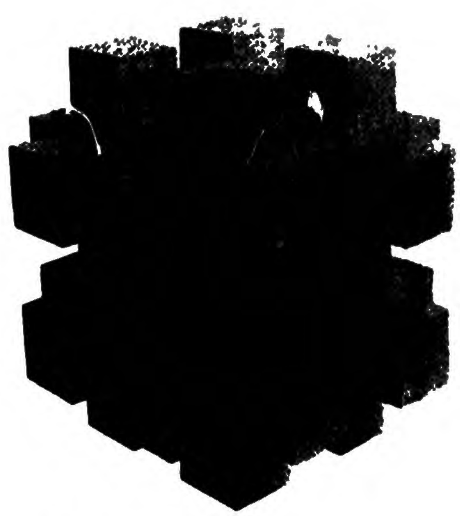
In the development in the style of several artists we can recognize a trend towards more abstraction. Well known are series of pictures of P. Mondriaan (The tree, pier and ocean) of Malevitz (from cubism towards suprematism) and others. We choose here as an illustration some examples from the 'oeuvre' of B. Nicholson, who is very explicit in the demonstration of the process of abstraction.

An essential feature of abstraction in mathematics is the fact that an abstract theory has many, quite different applications and interpretations as well in mathematics as in other fields. Abstraction in art sometimes leads also to polyinterpretability; we recall Miro's polite answers!

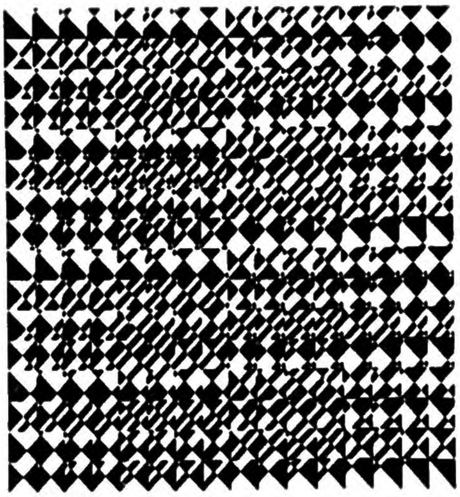
6. *Structure*

According to modern views, mathematics is no longer the science of number and space but nowadays it is the art of structures. Not only in philosophy but also in the visual arts we can recognize a trend that we can call structuralism.[4] The design of Peter Lowe 1, 2, 3, 4, ... at the head of this note can be called a work of structure. And there is structure in the space contrapositions of E. Hilgerman.



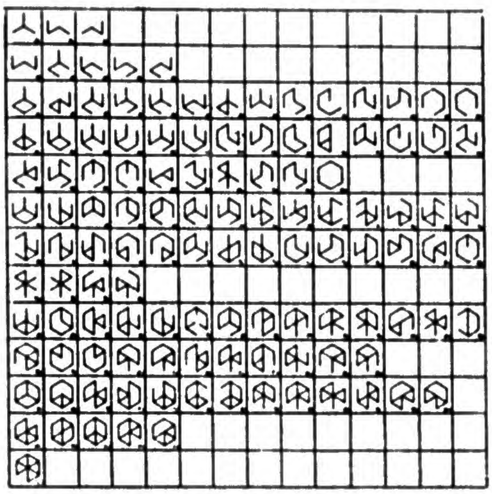


In the work of Roskam, there is the structure of the dyadic number system. The whole work of art is only a visualisation of the dyadic counting 1, 10, 11, 100, 101, 110, 111, and so on!



Almost all computer art[5] uses recursive structures, and it tells us stories of recursieve nature. We could distinct between art generated by a computer and art made by using a computer as aid in the process of the design. Of course there is a close connection between this structural art and the design of ornaments.

One of the most consequent structuralistic artists of our time is Sol LeWitt. His incomplete open cubes are just a mathematical problem. Construct all connected subsets of the edges of a cube. The problem is



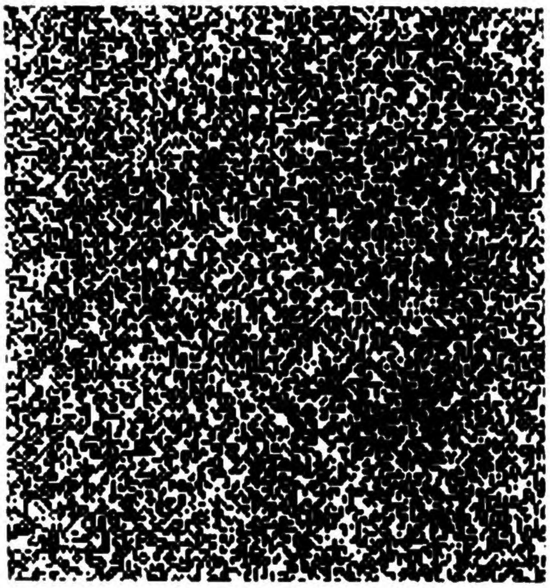
identification under congruence. Sol LeWitt not only makes a list of these, but he also constructs these out of wooden sticks. So there is need for a whole room in the exhibition for one piece of art. And the catalogue

of these incomplete open cubes is quite a book with photographs of all of them.

7. *Stochastic art*

In modern art stochastics is used, too. It is obvious that if you can use recursive structures, you can also use stochastic structures. You can do it in the way the Dutch artist Herman de Vries does. He mentions explicitly in a catalogue that he uses the tables of random numbers of R. A. Fisher and F. Yates (Statistical tables for biological, agricultural and medical research). Peter Struycken uses both recursive structures and stochastic processes. We give an illustration of a painting of Jean Arp: "Constellation according to the Laws of Change". In 1930 when

this painting was made one could not use random generators in computers. Morellet used a telephone directory as a source for a random process in: "Répartition aléatoire de 40.000 carrés suivant les chiffres pairs



et impairs d'un annuaire de téléphone". Another painting of Morellet is called: "Quatre répartitions aléatoires de deux carrés suivant les chiffres 31 – 41 – 59 – 26 – 53 – 58 – 97 – 93". The kinetic objects and the drawings of G. v. Graevenitz also use a stochastic ideom, but we only made an arbitrary selection in this field.

8. *Still more possibilities*

In this paragraph I mention some other possible headings for other chapters in this story. We had to come to an end, and then we remarked that we did not mention Vasarely, we did not give enough details on M. C. Escher, we should have given attention to the American artists like Noland, what about Kelly, and so on.

Still worse we gave no attention to the kinetic art,[5] to the art of equilibrium in the work of Kenneth Snelson. We did not use moirée, the wonderful phenomena of two pairs of silk stockings over a female leg.

We only give a photograph of a sculpture that demonstrates as well mathematical structure as the effect of moirée, it can be found on the beach in Ouchy, near Lausanne and was made by Duarte.

9. *Psychology*

Since we are together in a conference of the PME, we ought to give a lot of attention to the mathematical aspects of the theory of perception. But time has passed away. So there is no opportunity to discuss perspective in the light of the mathematical theory of visual perception. Neither much time is left to discuss the scientific background of the socalled "im-

possible figures" (Penrose). In the exhibition you can see work of the artist Reutersvärd created, long before the Penrose paper. Work of Escher using the impossible figure in his graphical work, and works of some

other artists using this phenomenon of human perception of objects, impossible according the mathematical laws with which we describe our surroundings. And still we can see them, although we know they cannot exist!

In the notes we give only a very restricted number of references:

1. H. Weyl: *Symmetry*, Princeton.
   A. V. Shubnikov and V. A. Koptsik: *Symmetry in Science and Art*, New York, 1974.
2. Pierre Descargues: *Perspective*, Paris, 1976.
   Jean François Nicéron: *La perspective curieuse ou magic artificielle des effects merveilleux de l'optique par la vision directe, la catoprique par la réflexion des miroirs plats, cylindriques et coniques ...*, Paris, 1638.
3. H. E. Timmerding: *Der Goldene Schnitt*, Leipzig, 1923.
   Luca Pacioli: *De divina Proportione*, Wien, 1889.

4. George Rickey: *Constructivism*, New York, 1967.
Abstraction, création: 1931–1936, Musée de l'Art Moderne de la Ville de Paris, 1978.
Systems, Arts Council, London, 1972.
5. *Cybernetic Serendipity*: Studio International, London, 1968.
H. W. Franke: *Computer Graphics*, Computer Art, London, 1971.
M. L. Prueitt: *Computer Graphics*, New York, 1975.
6. F. J. Malina: *Kinetic Art: Theory and Practice*, New York, 1974.
Frank Popper: *Die Kinetische Kunst*, Köln, 1975.

RÉGINE DOUADY

THE INTERPLAY BETWEEN DIFFERENT SETTINGS.TOOL-OBJECT DIALECTIC IN THE EXTENSION OF MATHEMATICAL ABILITY

Examples from Elementary School Teaching

INTRODUCTION

We are concerned with the process which leads to the acquisition of mathematical knowledge in class situations.

As an experimental field we have chosen the five years of French elementary school (age 6 to 11). Twenty pupils have been observed throughout the course of 5 years, others for part of this course.

We shall make some hypotheses concerning the acquisition process. These hypotheses take a meaning only in their realization through specific teaching projects. Therefore, we had to design and implement a teaching organization in accordance with our hypotheses, and test its impact during the actual process.

For the content, the project is centered on the measure of length and area (mainly of rectangles). It includes the learning of decimal numbers. We emphasize the intricate though distinct roles of measure and number by introducing functions, in both their algebraic setting and the graphic setting. These act as auxiliary settings, interaction with geometry and numbers.

Some conditions were particular:

– the teachers could follow their pupils one for the first 3 years, one for the two last ones. They were particularly good.

– the pupils had 40 minutes of study after class to do their homework (French and mathematics) under the control of the teacher.

– the classroom opened directly on the gymcourt, and this allowed some freedom with schedules.

However, the school was located in a standard suburban area and the pupils were recruited according to their residence only (as always in France). The school was 'école d'application' without special experimental status. Taking into account the usual practice, the project was initially planned for 4 or 5 years. Actually, most objectives were completed at the end of the third year, including the learning of decimal numbers.

Thanks to this unexpected timing, during the two last years, we could test the robustness of the learning and the evolution of the pupils' conceptions.

This acceleration of the learning cannot be explained only by the good working conditions. We think that we have acted on some control variables of the learning process.

OUR PROPOSALS

Usual pedagogy uses mainly the 'learn-apply' method, we note two points to be contested:

(1) Problems seldom involve the properties of concepts which actually justify their use in science. For instance, rarely in elementary school are decimal numbers used to designate, with an arbitrary good approximation, a measure which cannot be designated exactly by a number.

(2) Usually, concepts are presented in one setting and the application required stay in it.

We propose another way of organizing the teaching. It relies on the *Tool-Object-Dialectic* to recover the meaning, and on *Interplay between different Settings* to induce 'déséquilibres-rééquilibration', defined later. There, *Problems* will play an essential role.

Theoretical Frame

Our proposals depend on an outlook on mathematics, and rely on many previous studies concerning learning process; both are described below.

Concerning the Learning Process

From the works of Piaget, Vergnaud, the Geneva School of Social Psychology we remember the importance of the action, the role of 'déséquilibres-rééquilibration', the role of socio-cognitive conflicts. Concerning the importance of formulation and proof we refer to G. Brousseau, C. Laborde, and N. Balacheff.

We also take in account the necessary existence of a *didactical contract*, a phenomenon well studied by G. Brousseau.

Concerning Mathematics

In their research, mathematicians generally face problems nobody knows how to solve (on the other hand, pupils face problems which they believe their teacher can solve). To solve their problems, mathematicians are led to create concepts as tools. In order to be conveyed to the scientific com-

munity, these tools are taken out of their context and assume the characteristic of an object. It also happens that mathematicians create object directly in the process of organizing a branch of mathematics.

Thus, we say that a mathematical concept is a *tool* when our interest is focussed on the use to which it is put in solving problems. *By object* we mean the cultural object which has a place in the body of scientific knowledge, at a given time, and which is socially recognized. We consider as an object every mathematical notion presented by its definition, and possibly examples, counterexamples, structural description.

A concept is used as a tool by a pupil if it allows him to tackle problems. It may be either implicit or explicit. It is *implicit* if the pupil cannot justify his procedure without referring to notions he is actually not able to formulate (or he may formulate only in terms of action in a particular context). It is *explicit* when the pupils formulate and justify the notions they use. As an example of a concept used as a tool, consider the following problem – Is there a square with area 12 cm^2 ? – and this answer from a pupil: "for a square of side 3 cm the area is 9 cm^2, for 4 cm it's 16 cm^2, when the side goes from 3 cm to 4 cm there is a moment where the area is 12 cm^2". We recognize the relation 3 cm $\to$ 9 cm^2, 4 cm $\to$ 16 cm^2 as an explicit tool. Moreover, the numerical function $x \to x^2$, its continuity, the intermediate value theorem are all needed to make argument precise. All that is used as implicit tools.

Tool-Object Dialectic (T.O.D.)

We call Tool-Object Dialectic the following process, where we can distinguish six phases:

A problem is given to the pupils, which has a meaning for them. With their knowledge, they can begin to tackle it, but they cannot solve it complete. The concepts which the learning process is aimed at are the tools suitable for the question.

EXAMPLES: (1) How to determine and calculate the area of various rectangles which have the same perimeter?

This is an interesting problem for pupils who know how to answer if the measures are integers, who are convinced that every rectangle has an area, but who do not know what to do when measures are not integers.

(2) Partition: colouring a grid in 3 colours.

A point with coordinates (a, b) represents a rectangle R with sides of lengths a and b. The problem is to compare the measure of the area $A(R)$

with a chosen value k. More precisely the rule is, for $k = 24$:

- if $A(R) < 24$ colour the corresponding point in blue;
- if $A(R) > 24$ colour it in red;
- if $A(R) = 24$ colour it in black.

Is there a square among the rectangles?

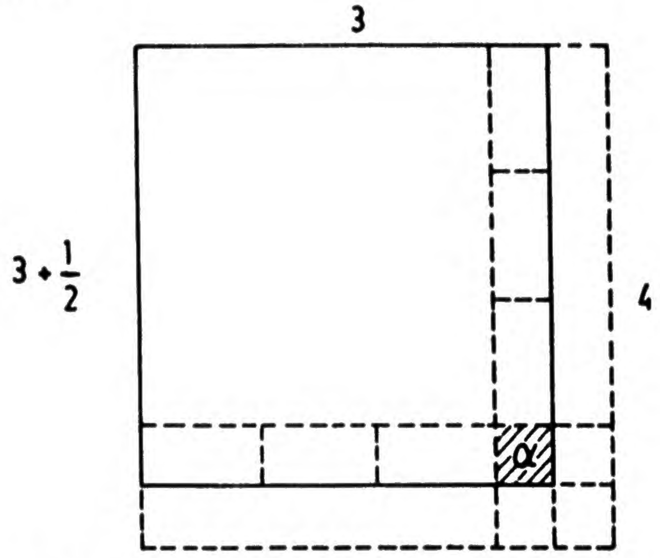
T.O.D. phase a: 'The Explicit Old ...'

Mathematical concepts are implemented as explicit tools to begin to solve the problem.

In the examples mentioned above, pupils may suggest several rectangles and calculate the area using only integers (it depends on the numerical values involved). In the second problem, they can pick a point on the grid, read the coordinates, a, b, multiply $a \times b$ compare to 24 and colour it.

T.O.D. Phase b: Research 'The Implicit New'

Difficulties are met in solving the problem completely either because the primitive strategy is too laborious (many operations, possibly errors...) or it no longer works. Pupils must look for other means better adapted to the situation.



We come back on the first example above. If the perimeter is 16 cm, pupils must find two numbers a and b so that $a + b = 8$. Among these rectanges, they find the square (side of length 4 cm, area 16 cm^2). If the perimeter is 14 cm, pupils may find $(3 + ½) + (3 + ½) = 7$ and meet a difficulty to determine the area of the square with sides of length $(3 + ½)$ cm. In this situation, a phase of research begins for the pupils faced with this problem. They are led to modify their practice (which was

to make a multiplication of integers) and adapt their knowledge. Here, they refer implicitly to additivity of areas and thus reduce the problem to the following: How to find the area of the small square? Pupils are led to name the different pieces, consider D as a part of the unit square and exploit the fact that they can pave the square with four copies of D. Transferring this remark to a numerical formulation they obtain $4 \times a = 1$. The answer is $a = 1/4$.

More generally they are led to work the situation to extend the correspondence length → area to non integer measures. Numerically, they are led to extend the multiplication to rational numbers. The question is: how and with which meaning? Pupils will answer by doing an interplay between the geometrical setting and the numerical one as we have suggested above. We will come back on this later.

In the colouring problem, the procedure described (in phase a) allows to colour a few points (10 or 20). However pupils must find an economic algorithm to colour all the grid (thousand points or more). Actually they use the compatibility between order and multiplication to progress. This property is an implicit tool. It is implemented with integers, and it is extended to 'measures between integers' which pupils do not know how to designate. They declare:

– "A red point, more than 24, above a red point and on the right, even more, Red points".

– "Below a blue point and on the left, blue points".

– "Below a black point, blue points; above a black point, red points. Black points are more interesting than the others".

These are only the striking steps. Actually in the classes, they are separated by intense wading periods which vary a lot with the pupils and with the numerical values involved.

Later, pupils will look for the 'square 24' (that means the square with area 24 cm^2) either "at the crossing between rectangles with area 24 cm^2 and the squares" or numerically: what is the number x so that $x \times x = 24$? is-it possible to find such an x?

The problem is how to choose, designate, calculate with number near the cross, closer and closer. This research leads to decimal fractions to approach such an x.

T.O.D. Phase c: Making Explicit and Locally Institutionalized.

Some elements have had an important role in the previous phase and can be appropriate now by the pupils. They formulate these elements in terms of practice with their local use, or the teacher formulates them in terms

of objects. We call these '*new explicit tools*' with which pupils may become familiar.

Thus, the first problem above leads pupils to extend the multiplication to the fractions they know. For instance:

$$\tfrac{1}{2} \times \tfrac{1}{2} = x \qquad 4 \times x = 1 \qquad x = \tfrac{1}{4}$$

$$\tfrac{3}{8} \times \tfrac{5}{8} = (3 \times 5)x \qquad (8 \times 8)x = 1 \qquad x = \tfrac{1}{64}$$

$$\tfrac{3}{8} \times \tfrac{5}{8} = 15 \times \tfrac{1}{64} = \tfrac{15}{64}$$

This computation results from an interplay with the geometrical setting.

The second problem, involving the three geometrical, numerical, graphic settings, leads to another problem: how to find the black points?

This problem induces the pupils to give a meaning to a/b (where a and b are numbers, integers or no integers) through an interplay between the three above settings: that is the x such that $b \times x = a$.

This moment is convenient for developing the process of division.

The research of the square with given area 24 cm^2 or 27 cm^2 ... involves computation and comparison with various numbers and privilege decimal fraction $P/10$, $P/100$, $P/1000$ .... At that moment, an economical writing is needed. Pupils may propose several codes adapted to this need.

In this third phase, the works and the proposals of the pupils are discussed collectively. Sometimes, they solve the problem together. In 'situations de communication', knowledge diffuses in various ways depending on the pupils. Each one is not involved in the same way in the solution of the problem and in the use of the tools. That is necessary to establish a common understanding within the class. It will allow each pupil to have reference points in his mathematical knowledge. That is the aim of the following phase.

T.O.D. Phase d: Institutionalization – Status of Object

The teacher presents what it is new and is to be retained, with the usual conventions. He 'gives the course' presenting well organized definitions, theorems with proofs if necessary, emphasizing what is important in contrast to what is secondary. In our examples, that is the writing of decimal numbers, computation with them, comparison, their property to approach every measure as near as wished.

Thus, the teacher gives a status of object to the concepts used previous-

ly as tools. This new knowledge will work later as an old one, but the time has not come yet. Indeed, internalising the structure is extremely important in mathematics so that the knowledge is available. This internalisation has already begun in the former phases. To complete it, the pupil needs to test his (or her) knowledge working on his own in different situations. This is the aim of the following phases.

T.O.D. Phase e: Becoming Accustomed – Applying

The teacher asks the pupils to solve various problems or exercises which acquaint them with explicit tools recently institutionalized. In doing this, they may develop good habits and integrate common social knowledge with their own particular knowledge.

These problems, simple or complex, involve only acquired competence.

T.O.D. Phase $f = a$: Sophistication of the Task or New Problem

New knowledge is used in a more complex situation involving other concepts which may be known (but separately learned) or may need to be learned.

EXAMPLE: Could you find a rectangle such that:

– The half perimeter is 41 cm and the area 402 cm^2.

– The half perimeter is 39 cm and the area 402 cm^2.

From now on, the new knowledge becomes '*old*' and a new T.O.D. cycle can begin.

Remarks:

(1) Sometimes more than one cycle ($a, b, c, e = a$) is necessary before a complete T.O.D. cycle (a, b, c, d, e, f) could be developed.

(2) It happens that mathematical habits become concepts as objects only after a considerable time (even several years). It is the case of functions and graphical representations.

(3) Provided that enough knowledge is developed by T.O.D., other mathematical notions may be introduced in other ways: directly by the teacher or in textbooks.

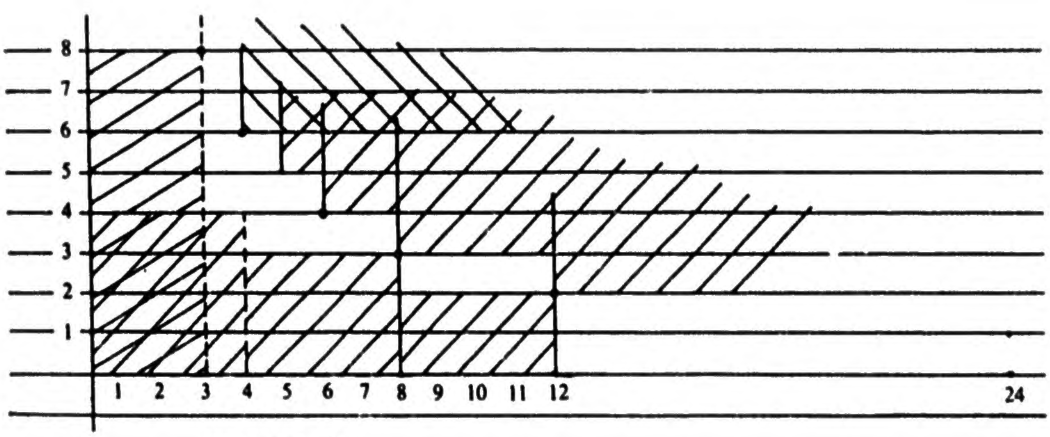
Interplay Between Settings (I.B.S.)

As a tool, a concept seldom stands alone, but rather it is part of a network of interrelated concepts. Mathematicians while researching for a solution to a problem spend a lot of time and energy interpreting their problems, looking for another formulation, transferring from one setting to another. Mathematicians use previously elaborated concepts as

experimental tools. Here, setting has its usual meaning as in the expressions arithmetic setting, algebraic setting, geometric setting, graphical setting … . Let us consider a *setting* as formed by mathematical objects, the relations between them, their different formulations and also by the mental images which are associated with these objects and relations.

By *changing setting*, different formulations can be obtained for one problem. The new formulations are not necessarily exactly similar, and so they offer a new approach to the difficulties of the problem and suggest the use of tools and techniques not available in the first formulation. This occurs if the signifiers involved in the problem are embodied as elements of a space provided with some structure. Graphic representation is an example. We will further discuss this point in the context of the learning process in school. In the transfer process, the mathematician is looking for reasonable conjectures and for staging points to build a framework for the proof of conjectures. It happens that counterexamples emphasizing obstructions oblige him or her to move intermediate points and even to discard the original conjecture. The translation from one setting to another often leads to unknown results, to new techniques and ultimately to the enrichment of the original setting as well as the auxiliary settings.

We Think That We Can Create a Similar Situation with Pupils in School



Here, k = 24.

Let us return to the colouring of the grid. The problem is stated in graphic terms. Each point of the grid stands for a rectangle. To colour a few points the procedure:

read the coordinates (a, b) calculate $a \times b$ compare with k

is good and requires only working with numbers. To colour all the points, it is convenient for the pupils to remember the geometric meaning of the problem. In this setting they know that, if a rectangle is included in another, its area is smaller. They can put this conviction to work to colour many points without computation. In doing this, they reduce the zone of uncertainty.
The black points are particularly interesting. *Problem*: How to find and locate them on the grid?

Recall that we discussed this problem said above (T.O.D. phase c). For a given numerical a, the other coordinate b is such that $a \times b = 24$. To locate the points (for various values of a) on the grid (graduated in 10th, even in 100th), pupils have to compare fractions $24/a$ with $P/10$'s or $P/100$'s. The division process has started.

Vice versa, by choosing a point on the uncoloured part of the grid between two black points, a pupil may read approximately the coordinates and is led to calculate with numbers he would not have chosen by himself, to make precise the colouring and the shape of the boundary. As they investigate the shape of the boundary, pupils are developing their knowledge and their competence about old and new numbers. They are developing also topological conceptions: neighbouring points, boundary points, boundary curve between two sets, subsets of the plane. Note that at the beginning, points of the plane were just stand-ins, symbols for rectangles. They have become the object of a geometric study. Their status has changed from signifiers to signified.

Thus, pupils may realize that even if they cannot produce exact solutions, it may be very reasonable to look for approximate ones, at the same time becoming conscious than an approximation is a reasonable answer. They will need several years to conceive of a real number as an infinite sequence of approximations. In their research of a square with area 24, by changing setting they are approximating $\sqrt{24}$.

By *Interplay Between Settings*, we mean changes of setting brought about by the teacher, within suitably chosen problems. In this process, we can distinguish three phases.

(1) *Transfer and Interpretation*

Pupils face a problem which is stated in one setting and which they cannot solve completely within this setting. But, thanks to their background and training, their analysis of the problem leads them to translate it in whole or in part into another setting, creating links between the various settings involved (between objects and between relations).

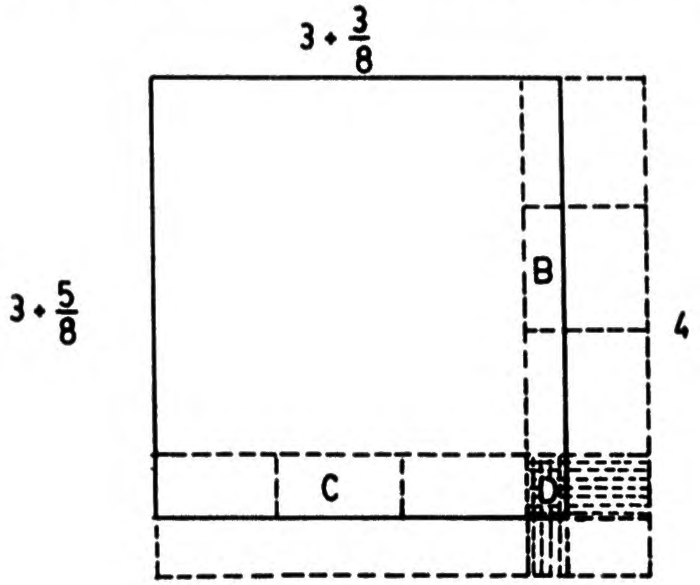
(2) *Imperfect Links*

However, the links between the settings are imperfect either for mathematical reasons ($a + b = 8$ admits negative solutions which do not represent measures of rectangles) or because the pupils knowledge is inadequate. This situation is a source of imbalance. As an example a pupil may be able to draw several rectangles with perimeter P (by deforming or compensating); however because they do not know enough numbers they cannot designate the measure of their sides, unless this measure is integer or a half integer. Moreover, conceptions and abilities of pupils vary not only from one setting to another but also from one pupil to another. One pupil may consider a square with arbitrary side-length from 3 cm to 4 cm, another will admit only those whose sides he can measure. One pupil will be able to calculate with halves and quarters, another will not. The correspondence with the numerical setting is defined only for some rectangles which are not the same for all the pupils in the class. These pupils who enjoy a strong conviction of the existence of rectangles regardless of their ability to designate the length of their sides will be imbalanced between their geometric conviction and their knowledge about numbers.

(3) *Improving the Links and Extending Knowledge*

The communication between settings allows the reequilibration. We have seen examples above: the colouring problem, the research of a square with area 24. Take another example: the research of the area of the rectangle $(3 + {}^{3}/_{8})$ cm, $(3 + {}^{5}/_{8})$ cm.

Most of the pupils distinguish four pieces in this rectangle. They are convinced that the area of the rectangle is the sum of the areas of the 4 pieces. This is clear geometrically. They can compute easily one of these



areas (3 cm × 3 cm = 9 cm^2). They have to compute the others. The problem is the following: how to determine the area of the rectangles (1 cm, $^3/_8$ cm), (1 cm, $^5/_8$ cm), ($^3/_8$ cm, $^5/_8$ cm).

– Actually, they need the area $B + C$. They may join together the corresponding rectangles and obtain $B + C$ without computation.

– They cannot avoid the analysis of the drawing to obtain the last area D.

Pupils are led to look for a small piece to have both D and the unit square. The square with side-length $^1/_8$ cm answers to the question. The unit square requires 64 copies of them to be paved. Paving is a geometric operation. The numerical translation of the adequate pavings provides the desired areas through the numerical solution of equations ($64 \times x = 1$, $8 \times x = 1$).

Remark: we have tried to describe separately the T.O.D. and Interplay Between Setting. Actually, we saw how they are intimately related.

To illustrate T.O.D. and I.B.S. we chose examples from the learning process of decimal numbers we have elaborated. In the next paragraph, we give the main mathematical points on which it relies and the diagram of the various settings (with relations between them) required. We give also a possible diagram for designing a problem, and finally the standard planning of a session. Then, we will develop an example of 'didactic engineering' in the first year of curriculum (6 year olds) which lasted three weeks.

DIDACTIC ENGINEERING

(1) *Decimal Numbers: Various Points of View*

– From a topological point of view, problem-situations are studies within interplay between geometrical and numerical settings. Estimates from above and below, progressively improved, are the suitable tool.

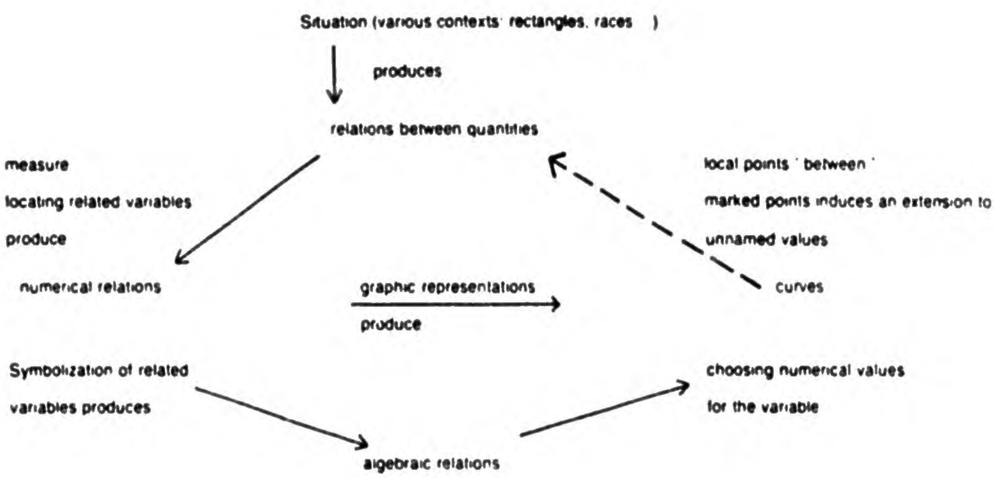
– From an algebraic point of view, the euclidean division is practiced in problems where the quotient and/or the remainder are to be used, also in isolated exercises either under the algebraic formulations $a = (b \times q) + r$ with $r < b$ or just as an operation

$$\begin{array}{r|l} a & b \\ \hline r & q \end{array}$$

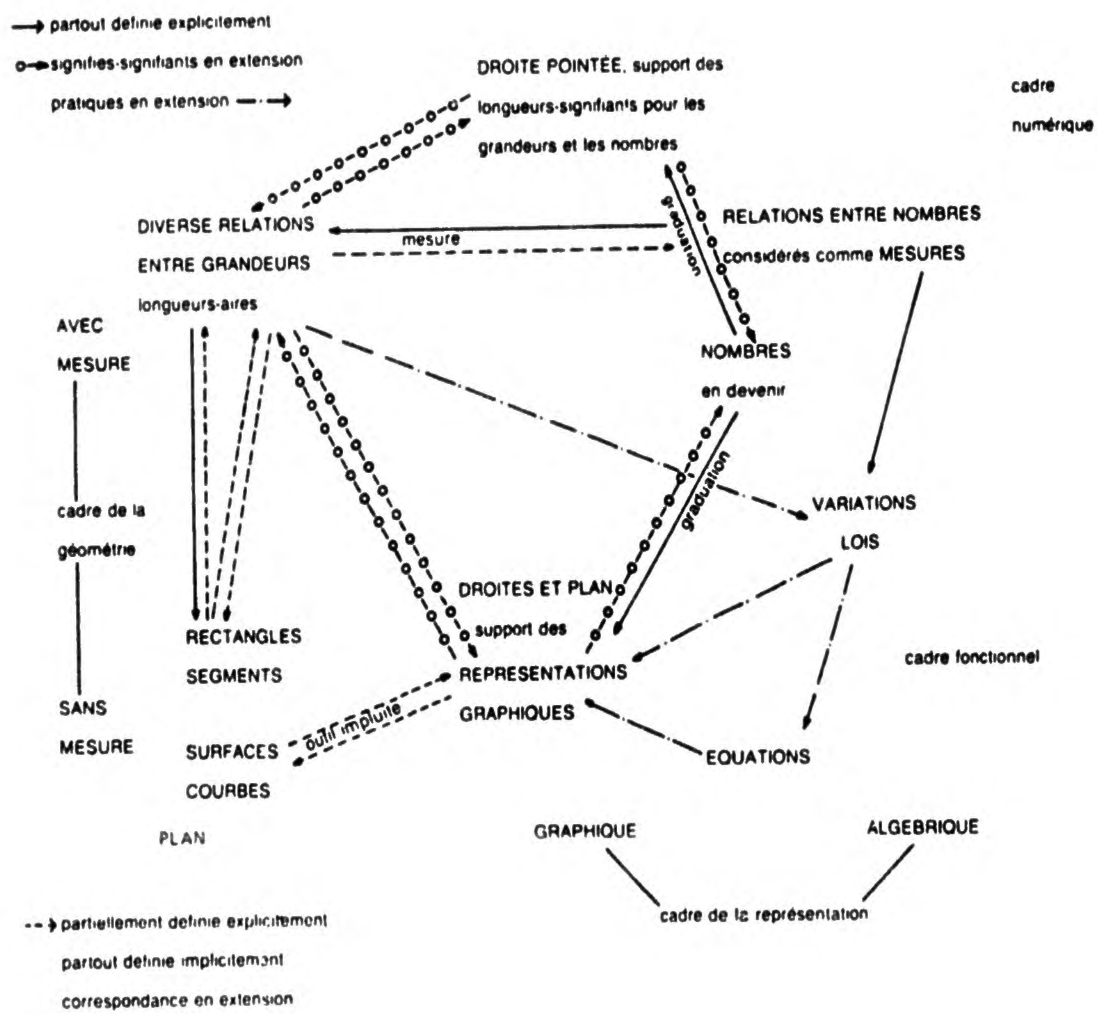
– From a strictly numerical point of view, oral computation is practiced daily. It involves explanations, justifying or possibly rejecting the various computing methods.

– Computation is involved in problems. There is no systematical training to written operations. However evaluation tests may require some of them.

– Functions appear in problems, as tools to work mainly with graphics involved in a double relay signified-signifiers-signified. Among them, linear functions are identified. Numbers, and particularly fractions or decimal numbers, are used to label linear functions and transforms computation with linear functions (addition, composition, comparison of functions) into computation with numbers (addition, multiplication, comparison of numbers). Nothing of that is institutionalized.



Control by computation (and acquaintance with it) together with graphic estimation induces an extension of relations


partout definie explicitement
signifies-signifiants en extension
pratiques en extension
DROITE POINTÉE, support des
longueurs-signifiants pour les
grandeurs et les nombres
cadre
numérique
DIVERSE RELATIONS
ENTRE GRANDEURS
longueurs-aires
mesure
graduation
RELATIONS ENTRE NOMBRES
considérés comme MESURES
AVEC
MESURE
NOMBRES
en devenir
cadre de la
géométrie
graduation
VARIATIONS
LOIS
DROITES ET PLAN
support des
RECTANGLES
SEGMENTS
REPRESENTATIONS
GRAPHIQUES
cadre fonctionnel
SANS
MESURE
SURFACES
COURBES
outil implicite
EQUATIONS
PLAN
GRAPHIQUE
ALGEBRIQUE
cadre de la représentation
partiellement definie explicitement
partout definie implicitement
correspondance en extension

(2) *Standard Planning or a Session*

– Oral computation including explanation of results (about 10 minutes).

– Review of previous sessions on the same theme, if any (10 minutes again, possibly more).

– New work of the day: it includes three stages.

(1) The teacher gives the instructions. A collective discussion about them (not about method of solution) will clear up any ambiguity and must allow the pupils to appropriate the problem.

(2) Pupils work (personally or by team). A taking stock may occur if several pupils are faced with the same difficulty.

(3) There is a collective taking stock of the results presented through a representative sample of individual work or by each team. The class compares the various productions and agrees on which is best, or at least agrees to disagree and to further study the problem.

A session ususally lasts $1H\frac{1}{2}$. The conclusion should not be skipped; it can be postponed till after a recess, eventhough that might imply the next session is shortened.

As an example of didactic engineering, we present the striking points of the situation called 'the target-game' completely analysed in ([2] Chapter III, Douady, R.).

(3) *The Target-Game*

The aim:

(a) Use numbers, order and addition as tools. In interaction with that, extend the domain of numbers on which they can operate and make more precise the meaning of their writing.

(b) Increase the complexity of the problems they can deal with. Linked with that, construct an algebraic language and design representations (numerical-tables, sticks, graphics ...) to deal with a great flow of information and solve problems for which they are necessary.

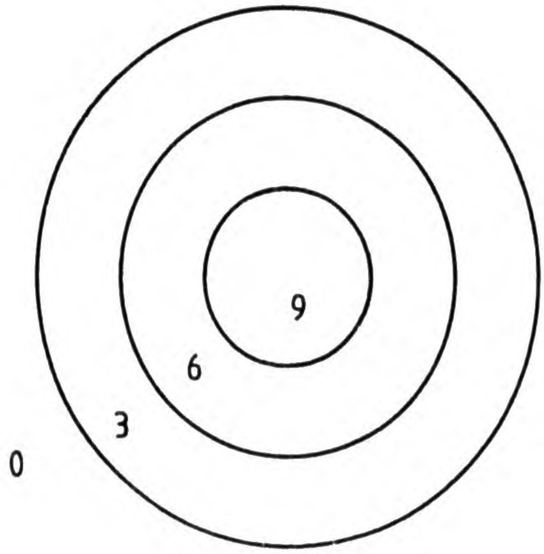
(c) Encourage the use of multiples and divisors in a simple situation. These notions are involved in numeration and their learning takes several years.

The Situation

It includes two types of games, with the same target.

– In the first type, players have to mark as many points as they can.

– In the second type, the game is to mark a number of points equal to a given number or as close to it as possible.



- The target has 4 zones marked 0, 3, 6, and 9 for the smallest.
- Every player receives a ball. He can throw it three times.
- The winner is the one who has the highest score.

Question: Arrange the players from highest scorer to lowest scorer.

T.O.D. Phase a: the Explicit Old ...

Pupils knew that to find their score, they had to add their marks, i.e. each had to add three numbers smaller than 9. They had the technical competence to do that. Then together they had to arrange 28 numbers (from 28 players) smaller than 27 from the highest to the smallest. They were able to do that. The only problem was to know how to be sure that they had the right information. After discussing it, they decided to each write his 3 marks beside his name.

T.O.D. Phase b: ... and the Implicit New (Research)

Players are grouped in teams of 4. Each one can throw the ball three times. The score of a team is obtained from the points of all the players of the team. The team who has the highest score wins.

Question: Arrange the teams from highest to lowest.

Pupils know they have to add marks of each player from the team. But even if they played badly, twelve numbers is to many for most of them to add. So, to compare scores they have to use another procedure.

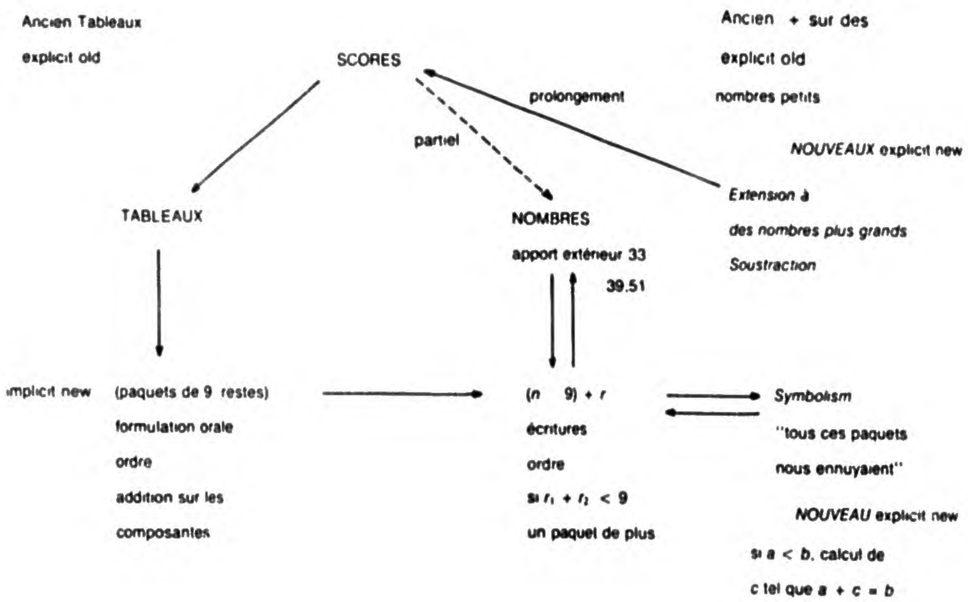
Evolving *new encodings* of the score:

To have the right information about the game, teams need a written record of their marks. They decide to write them in a table with 4 lines (4 players by team) and 3 columns (3 balls per player).

Question: How to compare two such tables? How to recognize tied teams?

Contingency and uncertainty: if we want pupils to be active in their research, we must so arrange things that they need to analyse in detail the table of their team. This is possible if pupils may ask various questions about the game, the marks, the players, the rules of the game ... before the teacher asks them to order the teams. In the classroom observed, pupils rearranged the players of a same team, looked for the tied players, players who had three times the same mark ... .

This phase is important: the pupils need the freedom of action to have a real choice of a procedure and to put implicit tools to work.



Interplay Between Game Encoding and Numbers

Later, to compare the tables, they focussed on the number of 0's in a team and the number of 9's. They convinced themselves that they could change the numbers within a table without changing the value of the table, i.e. the score: they argued that $6 + 3 = 9$ or $3 + 3 + 3 = 9$. In-

stead of one 6 and one 3 or three 3's they put one 9 and one or two zeros ($6 + 3 = 9 + 0$, $3 + 3 + 3 = 9 + 0 + 0$) to have always a table with twelve places. They would have been able as well to put two 9's instead of three 6's, but they never had to do it.

Pupils' *claim*: the team which has the most 9's wins.

New problem: For each team, count the 9's.

Some 9's are already there, but others need to be 'made'. In this connexion, a misunderstanding arose between the pupils and the teacher. The pupils faced with the following table:

9	9	6
6	6	9
0	0	0
3	3	6

said "with two 6's we make a 9". For the teacher, 6's and 9's are numbers and $6 + 6 = 9 + 3$. She was trying to extract this answer from the pupils. She knew they had the competence and she did not want to give them the answer. Various pupils suggested some modifications in 6_6 in order for a 9 to appear. The situation stayed jammed twenty minutes until a pupil understood that the teacher wanted a numerical equality. He said "I have thought a little bit" and wrote $6 = 3 + 3$. After that, all the required 9's were obtained quickly.

We interprete this jam in the following way: both the teacher and the pupils work in two settings: the pupils in the game and in the code (the situation did not require more, at this point), the teacher thinking of her teaching in the code and in numbers; but they share only the encoding setting. The unjamming occurs only when one pupil enters the numerical setting. A sign of this is that he is immediately followed by the entire class.

New formulation: pupils consider couples ($n \,.\, 9, r$). Orally they speak about "packs of 9's and remainder". They write (after various proposals and discussion) $n \square 9 + r$ ('$\square$' means packs). This new writing is a progress only if it helps them to order the couples. In terms of the game they have convinced themselves that the team with the most of 9's won, without considering the remainder. Actually, we observe that they use the following algorithm:

$$\text{if } n_i < n_j \qquad n_i \square 9 + r_i < n_j \square 9 + r_j$$

$$\left.\begin{array}{r}\text{if } n_i = n_j \\ \text{and } r_i < r_j\end{array}\right\} \; n_i \square 9 + r_i < n_i \square 9 + r_j$$

This discovery on alphabetic order of couples is meaningful for the pupils only in terms of the game. It is an implicit tool and will become a practice to compare numbers written $n \square 9 + r$.

T.O.D. Phase c: Making Explicit

Question: in another classroom, they played at the same game. One team had 39 points. Where is this score located among ours?

Pupils had to compare scores written such as $n \square 9 + r$ and 39. They had two possibilities:

- write 39 as n '$\square$ 9 + r'.
- Compute the score of the various teams.

In both cases, they must use the meaning of the writing: $39 = 3 \square 10 + 9$.

Remark: at this point, we do not institutionalize anything. Pupils work to become acquainted with numbers written in this form.

T.O.D. Phase e: Becoming Accustomed with Scores Written $(n \square 9) + r$, in Particular Additions and Comparisons of such Scores

SITUATION. Each team plays a second game.

(1) Order the teams from loser to winner.

(2) Each team compares its score in the first and second game, and tries to discover how much better or worse it did.

The first task only involves previously acquired knowledge. It is analogous to a problem which has already been solved.

For some teams, the second task will involve subtractions with carries; if necessary, they will use sticks to solve the problem.

T.O.D. Phase f = a: Order All the Scores, and Find Out How Many Points Each Team Got in the First Game, the Second Game, and in All

The object is to reuse the tools the pupils have already developed, in a more complicated situation.

The first task is to find the total score of each team. The algorithm actually used by the pupils is:

scores $n \square 9 + r$ and $n' \square 9 + r'$ total score
$(n + n')9 + (r + r')$
you get an extra 9 if $r + r' > 9$.

Using this rule they can order the total scores.

T.O.D. Phase b and c: The Second Task is to Transform Score into Numbers

The pupils say "to count the points, we add up the 9's and the remainder". This means "group the points by 10 rather than 9". Several methods are suggested and explicited.

T.O.D. Phase d: Institutionalization

(1) Writing numbers requiring 2 or 3 digits in base 10, including the table:

hundreds	tens	units

(2) If $A < B$ then you can find C such that $A + C = B$.

T.O.D. Phase e: Becoming Accustomed

With (2) above, for vari)us numerical examples; in each case, calculate C.

T.O.D. Phase f: New Situation

Reuse of substraction as above, now as an explicit tool. This will be done in the context of a new game, played with the same teams, using the following rule:

(1) Each player throws the ball once.
Team which gets 18 points wins.
(2) Each player throws three times.

Any team which gets 50 points or a number as near as possible wins.

Conclusion

Our project was to test cognitive hypotheses about the learning of mathematics in the classroom. In order to do this, we have reconsidered the way the mathematical knowledge we had to teach was cut up, and its organization in learning sequences. The long period of time available to us gave us the freedom required to implement interplay between settings, to use the mathematical tools to be learned with their implicit-explicit character, and also to create habits. Note that computers provide a new frame of work which may turn out to be efficient if it is actually involved in interplays of settings, but not if it stays aside without interaction with the other settings.

However, in order for the dialectic process we propose to be set in mo-

tion, thresholds of the two following types must be respected:

– Pupils need to have a 'critical mass' of knowledge, different in various settings

– Pupils should be asked serious problems: there is a threshold in questioning beneath which problems do not set off the process which leads to lasting knowledge.

Moreover, Tool-Object Dialectic and Interplay Between Settings may allow teachers and even more those who teach teachers to elaborate teaching methods and analyse the problems they encounter.

There is still the crucial point of how to organize the teaching in time. Indeed, the time spent floundering seems a waste of time in the short term. However, in the long run, the knowledge thus acquired has a more stable foundation and is better adapted to further use and modification.

The pupils need not acquire the whole of their mathematical knowledge through this dialectic process; however, the basic framework should be acquired in this way. This basic frame work can only be elaborated over several years. To carry out this teaching over many years, curriculum will need to be adapted (as is actually happening in France) and teachers will need to coordinate their teaching much more carefully than at present. Perhaps the introduction of computers, which is in any case bringing new flexibility, is a good occasion for such changes.

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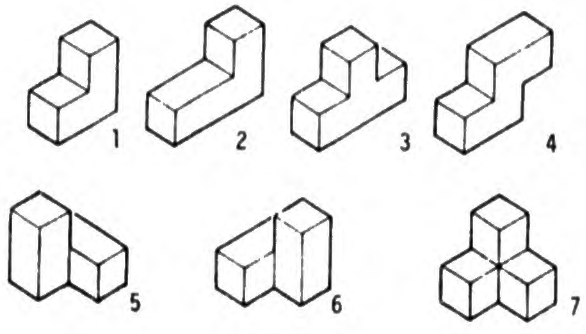
CLAUDE GAULIN

THE NEED FOR EMPHASIZING VARIOUS GRAPHICAL REPRESENTATIONS OF 3-DIMENSIONAL SHAPES AND RELATIONS

INTRODUCTION

From 1978 to 1980, materials were developed at Laval University (Gaulin *et al.*, 1980) to serve as a basis for two courses called 'Explorations Géométriques' which have been offered since as part of an important distance in-service teacher education program in mathematics (PPMM) for elementary teachers of the Greater Québec area. Following guidelines the author had suggested earlier (Gaulin, 1974), a particular emphasis was put in those materials on activities intended to foster the development of spatial visualization and geometrical intuition. Several units were also devoted to the exploration of various types of (two-dimensional) graphical representations of polycubical solids. A few examples are sketched below.

EXAMPLE 1: As is well known, the following seven polycubical shapes can be put together to form a $3 \times 3 \times 3$ cube (SOMA cube puzzle, invented by Piet Hein).



After a few exploratory activities, students are shown SOMA cubes already constructed and asked: *Suppose that after long pains you have finally succeeded in assembling the seven pieces into a cube. How could you record your solution on a sheet of paper so that you may easily reconstruct the cube at any time you wish in the future?* This problem

always stimulates the production of a great diversity of types of plane representations of the SOMA cube. Here are a few examples (some drawings are reproduced from Wilson, 1973).

2	2	2
4	1	2
3	3	3

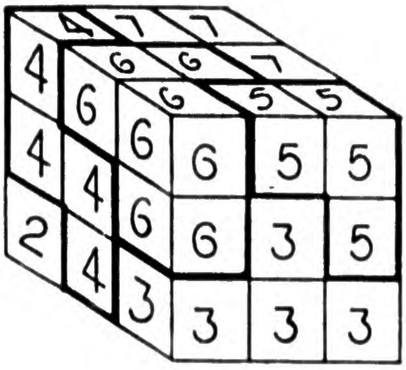
Bottom layer

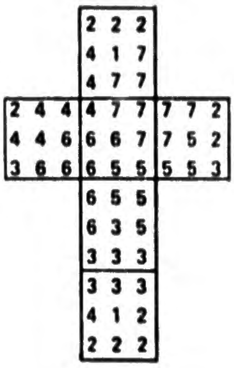
4	1	7
4	1	5
6	3	5

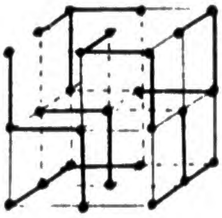
Middle layer

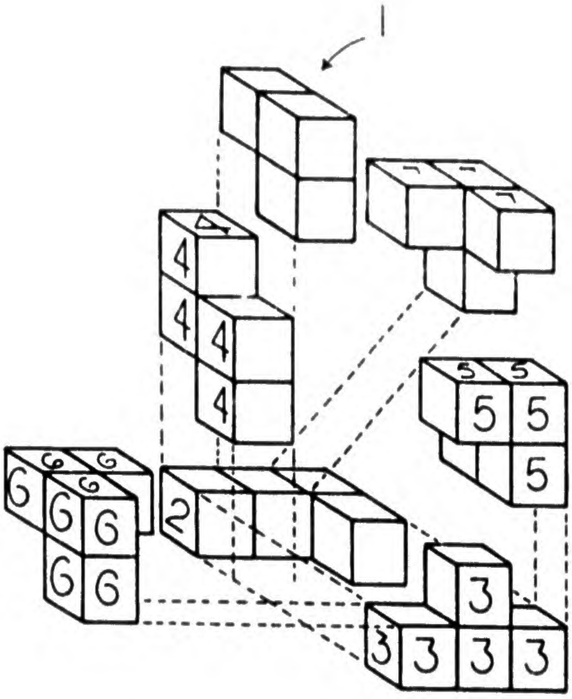
4	7	7
6	6	7
6	5	5

Top layer









#1	2,1,2	2,2,2	1,2,2	
#2	1,1,1	1,1,2	1,1,3	1,2,3
#3	1,3,1	1,3,2	1,3,3	2,3,2
#4	1,2,1	2,2,1	2,1,1	3,1,1
#5	2,2,3	2,3,3	3,3,3	3,3,2
#6	2,3,1	3,3,1	3,2,1	3,2,2
#7	3,1,2	3,1,3	3,1,3	3,2,3

Stories coded I,II,III from bottom

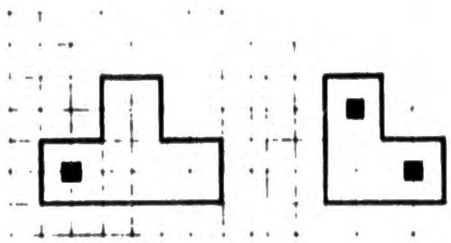
1	2	3
4	5	6
7	8	9

Coding on each story

Piece 1:	I - 5	II - 4	II - 5	
Piece 2:	I - 1	I - 2	I - 7	
Piece 3:	I - 3	I - 6	II - 6	
Piece 4:	I - 8	II - 7	II - 8	III - 7
Piece 5:	II - 2	II - 3	III - 3	III - 6
Piece 6:	II - 9	III - 9	III - 8	III - 5
Piece 7:	II - 1	III - 1	III - 2	III - 4

EXAMPLE 2: Students are given the topographical map of a familiar area near Quebec City (see page 57). Various questions are asked which require the visualization of peaks, valleys, steep portions of some trails, etc. by means of countour lines and other codes. One of the questions reads: *Imagine that you take a walk from LA DETENTE and that you suddently discover water springing from the ground at the location marked X. Try to draw on the map the path followed by the water. Compare your answers with those of your neighbours and try to agree with them about the path which is the most likely* ... (Subsequently, students are asked to answer the same question supposing that water springs at location 0 instead of X.)

EXAMPLE 3: A whole unit is devoted to the representation of polycubicial solids by means of orthogonal projections. In order to make the representation univocal, it is generally necessary to supplement the front or top or side views with more information conveyed via some code. Instead of readily introducing the well known standard code using continuous and dotted lines, we let the students first interpret various non-conventional codes and create their own! For example, using a set of congruent material cubes, they have to try and build the solid corresponding to the two given coded orthogonal views.

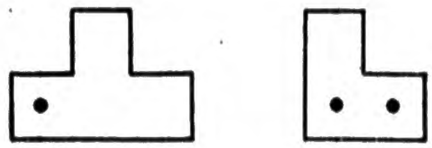


Legend

■ the cube you can see here is the second row

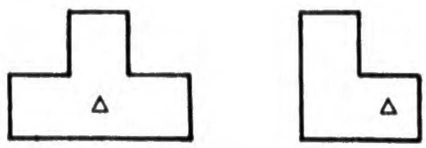
Legend

there is nothing behind the cube you can see here



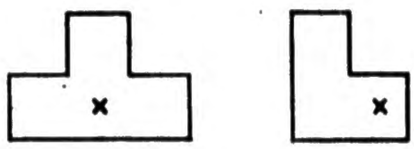
Legend

● there is at least one cube somewhere behind the one you can see here



Legend

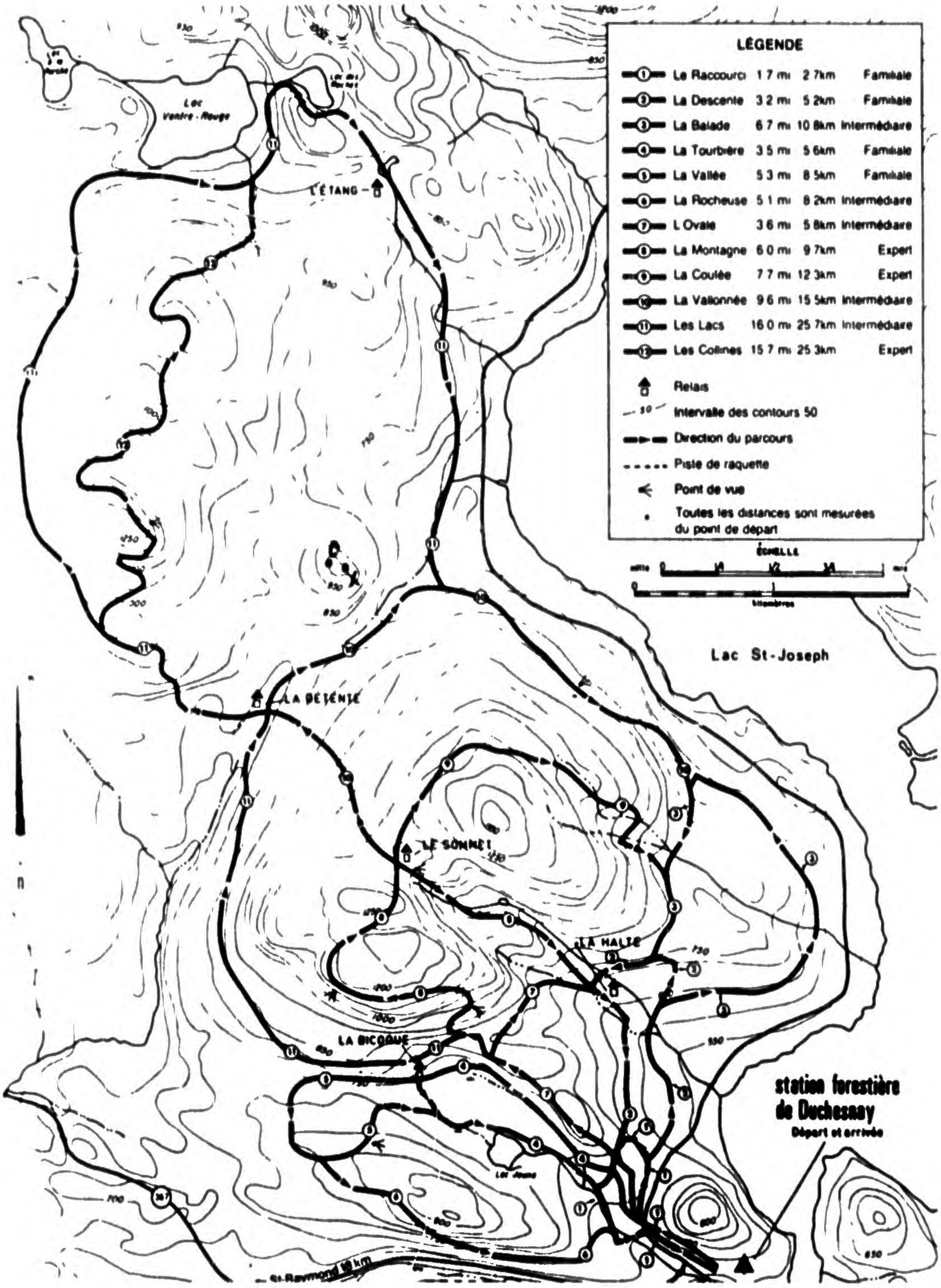
△ there is a cube immediately behind the one you can see here



Legend

× the cube you can see here touches at least two other cubes

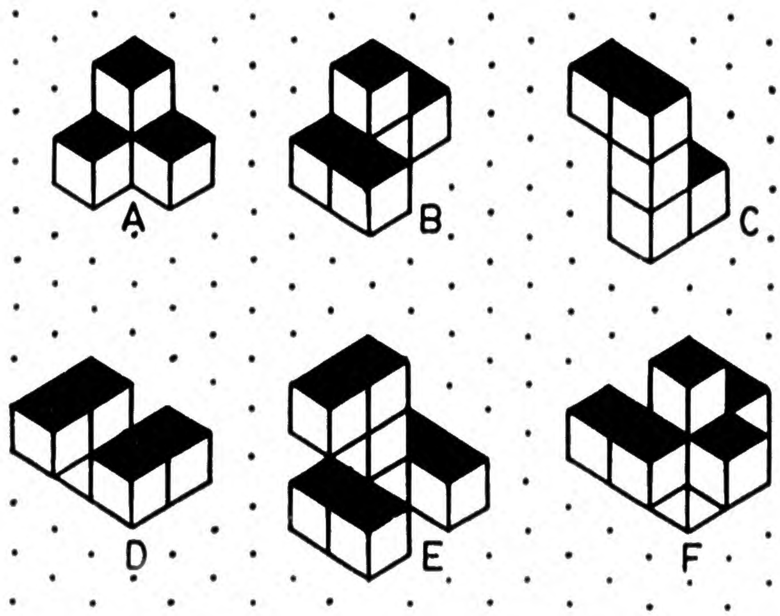
Of course, numerical codes are also used. Students also have a few opportunities to invent their own codes for representing other polycubical shapes by means of two or three orthogonal views. Subsequently, they become acquainted with the use of dotted lines in this type of representation via the use of the very interesting French game called STRUCTURO. However, no attempt is ever made to systematically develop techniques usually taught in technical drawing courses.



Although such activities involved actual experiments with physical materials, many teachers taking the courses experienced great difficulties in making and interpreting such graphical representations, which of course tended to hinder their ability to visualize the corresponding three-dimensional shapes and relations. Notwithstanding several attempts to improve those units, such difficulties persisted. This aroused the interest of a few PPMM collaborators, who decided to more closely investigate some problems related to the use of *plane representations to communicate spatial information*.

SPONTANEOUS REPRESENTATIONS ON PAPER OF 3-D SHAPES BY PUPILS

As a preliminary step, two exploratory studies (Gaulin and Puchalska, 1983) were conducted during 1982–83, with a number of about 500 pupils about equally distributed among 4th grade (10–11 years old), 6th grade (12–13 years old), 7th grade (13–14 years old), 9th grade (15–16



years old) and 11th grade (17–18 years old). The following task was administered in each of the 21 classes.

Every subject received *one* geometrical solid made of congruent plastic cubes glued together. The six shapes illustrated above were distributed about equally among the pupils of each class.

In addition to one plastic shape, each subject was given a sheet of paper with a brief instruction and plenty of space to answer. Half of the pupils got the following ('algorithmic') instruction:

Imagine that one of your friends lives in France and that he has got a whole box of plastic cubes, all of the same size. Now you would like your friend to build a shape like the one you have in front of you. Prepare a message you could send him so that he can build it. You may give your explanations using words or drawings, as you wish.

while the other half got the following ('descriptive') instruction:

Imagine that one of your friends lives in Ottawa. Near his home, there is a store selling all kinds of shapes made of plastic cubes. Now you would like your friends to go to that store and buy you one shape like the one you have in front of you. Prepare a message you could send him so that he can recognize that shape. You may give your explanations using words or drawings, as you wish.

The type of communication task used (preparation of a 'message') had been inspired from the work of Guy Brousseau.

One of the objectives of the study was to observe the *types of production* (verbal or graphical or mixed) *spontaneously given by the subjects.* The first striking observation was the *great variety of types of productions* obtained at all ages. (The other major observation – *the predominance of 'coded orthogonal views'* – will be commented during the oral presentation, if time allows.) Here are a few examples.

VERBAL DESCRIPTIONS

La figure que je vais décrire a 4 cubes, 24 faces, 36 arêtes, 21 coins et elle est rouge.

(10-year old Shape A)

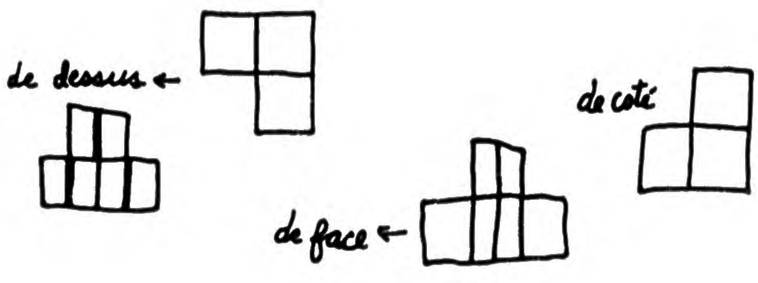
Place un cube à droite. Place un autre cube à gauche et un autre cube au centre. Place un autre en haut du centre.

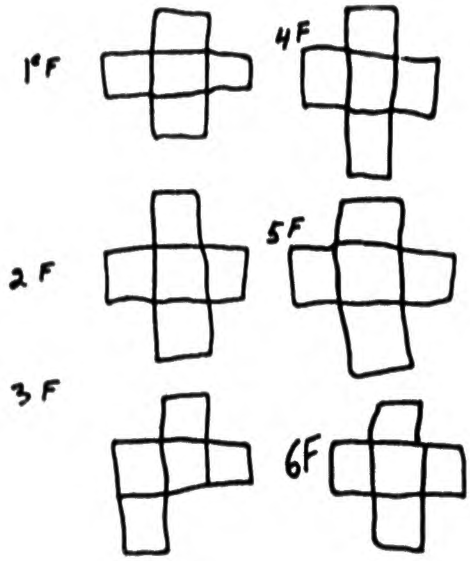
(10-year old Shape B)

Tu dois avoir une croix, avec un bloc sur le dessus au milieu. Il y a aussi un bloc sur l'autre face de la croix, mais il est placé par en-haut.

(9-year old Shape F)

SIDE VIEWS

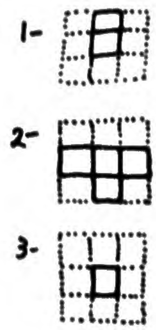




(9-year old Shape F)

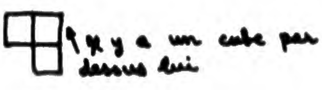
DESCRIPTIONS BY LAYERS

(12-year old Shape A)

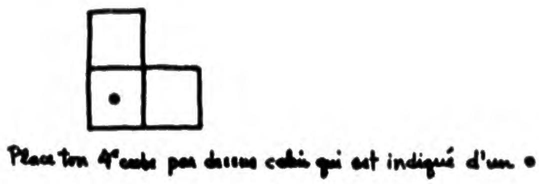


(13-year old Shape F)

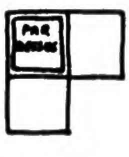
"CODED ORTHOGONAL VIEWS"



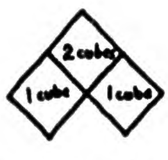
(12-year old Shape A)



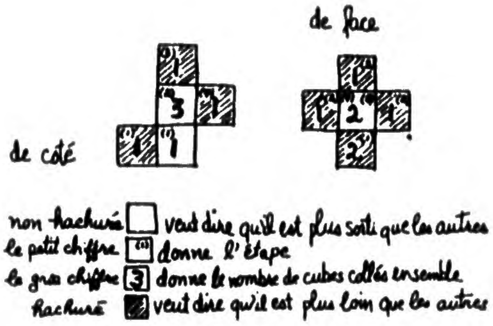
(14-year old Shape A)



(13-year old Shape A)

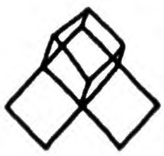


(13-year old Shape A)



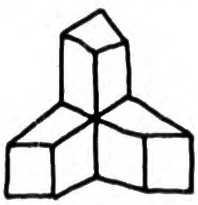
(16-year old Shape F)

ATTEMPTS AT PERSPECTIVE DRAWINGS

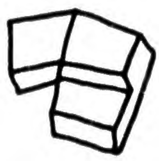


(9-year old Shape A)

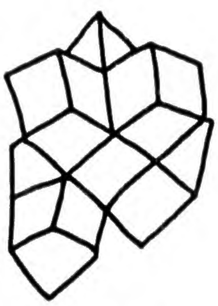
(13-year old Shape A)



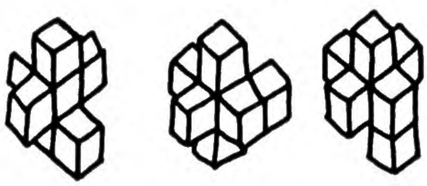
(13-year old Shape A)



(13-year old Shape A)

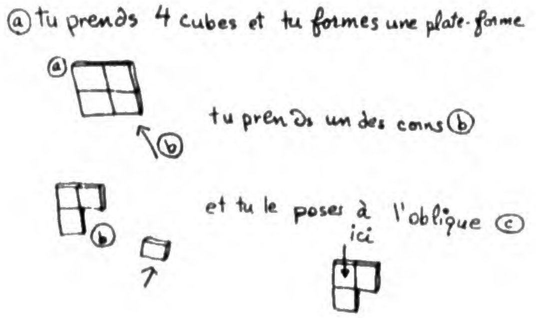


(10-year old Shape F)



(14-year old Shape F)

OTHER TYPES OF MESSAGES



(14-year old Shape A)

Remark: At the time the above exploratory studies were about to be completed, we were informed that a communication task of a similar type had been used, although with a different methodology, during experiments conducted by Bessot and Eberhard (1982a, b) in Grenoble. These French researchers also found a great variety of types of messages spontaneoulsy produced by elementary pupils. Franchi and de AzeveDo (1983) later made a similar observation in Sao Paulo.

The main contention of this presentation is that in mathematics education much more emphasis should be put on various types of plane representations of three-dimensional shapes and relations, both in the curricula of a majority of countries and in research and development.

REASONS FOR EMPHASIZING A VARIETY OF GRAPHICAL REPRESENTATIONS OF SPATIAL SHAPES AND RELATIONS

1. *Graphicacy as a Basic Educational Objective*

The ability to use various types of graphical representations is one aspect, among others, of what is sometimes called '*graphicacy*' in England, i.e. "*the communication of spatial information that cannot be conveyed adequately by verbal or numerical means*" (Balchin, 1972).

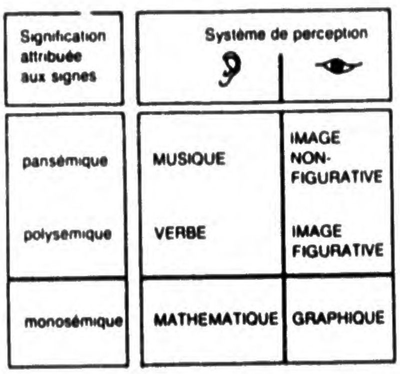
"Balchin and Coleman describe *literacy, numeracy, articulacy* (subsequently superseded by *oracy*) and *graphicacy* as the four 'acies', or 'aces' for short [...] Neither words nor numbers are superior or inferior as modes of communication. They are only more suitable or less suitable for particular purposes, and each ranges from the very simple to the extremely complex. They complement each other and achieve their highest level of communication when properly integrated. [...] Balchin argued that the discussion ought to be more concerned with modes of communication and that the three Rs should be replaced by the 'four aces'. He also noted that in France a similar discussion was taking place around the concept of 'four languages' in communication skills and the need to teach them to all

pupils. These four languages corresponded exactly with the four modes of communication distinguished in Britain.

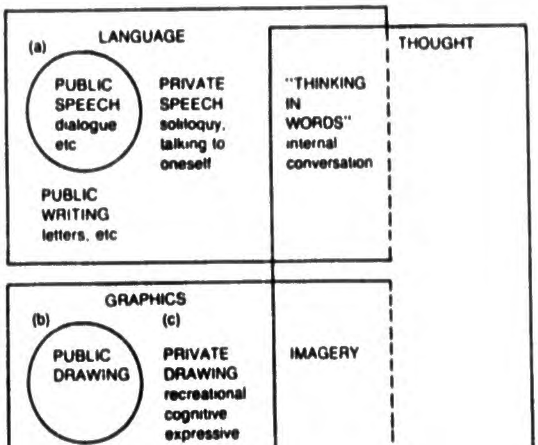
Meanwhile it has been argued by Boardman (1976) that geography teachers share with their colleagues in other subjects, especially English and Mathematics, responsability for ensuring that graphicacy is developed by all pupils before they leave school ..." (Boardman, 1983, preface)

As a geography educator, Boardman speaks of graphicacy insisting particularly on maps (of all kinds) and photographs. However we feel that his arguments apply equally, perhaps even with greater strength and more relevance, when 'graphicacy' is defined in a more general way so as (1) *to include the communication of nonfigural information by means of diagrams, schemas and graphs having a low degree of 'iconicity', i.e. representing abstract relationships* (cf. Bertin, 1967, 1977; Herdeg, 1981); and (2) *to take also into account the non-social instrumental role of graphical representations for concept formation, problem solving and more generally organizing thought* (cf. Van Sommers, 1984; Biehler, 1982; Chernoff, 1978).

Developing graphicacy as a basic educational objective is clearly a *multidisciplinary* enterprise in which mathematics education has an important contribution to make, just like geography education, art education, language education, etc. Providing all pupils with opportunities to *explore a variety* of types of graphical representations of *spatial and geometrical* – as well as numerical, logical, statistical, etc. – *information* appears one of the features of such a contribution.



Bertin (1967, 1977)



Van Sommers (1984)

2. *The Ever Increasing Practical Importance of Such Representations*

Graphical representations of various types are commonly used in a great number of practical situations and disciplines for conveying spatial information. Here are a few examples:

maps, plans and sketches: topographical, geographical, geological, meteorological, architectural, for finding directions (e.g. in a shopping center), for trains / buses / subway / airlines networks, etc.

diagrams and flowcharts giving instructions: for assembling (e.g. a piece of furniture or parts of a construction kit), for operating a machine, for sewing or knitting or crocheting, etc.

scientific or technical descriptive drawings: in anatomy, botany, mechanics, engineering, ...; models of atoms and molecules, etc.

Taking into account the *pervasiveness of computerized graphical displays* in today's society, the ability to communicate with (as a minimum to interpret) such graphical representations is likely to be more and more needed in the near future, more particularly the ability to use 'coded graphical representations' (Gaulin and Puchalska, 1985).

3. *The Need of Re-Establishing the Development of Spatial Intuition as One Major Goal for Teaching Geometry*

For years mathematics educators like Bishop, Clements, Mitchelmore, Tahta and others have been advocating that one major goal for teaching geometry that has been overlooked during the 'new math' wave and ought to be re-established is the development of students' spatial intuition, including their ability to visualize and to communicate spatial information by various means. Taking into account the real mess existing concerning the definition of terms like 'spatial ability', 'spatial intuition', and 'visualization', we prefer to think of such an objective with reference to the two types of ability constructs that have been proposed by Alan Bishop (1980) in order "to help mathematics educators focus on relevant training and teaching research...":

(1) The ability for interpreting figural information (IFI). This ability involves understanding the visual representations and spatial vocabulary used in geometry works, graphs, charts, and diagrams of all types. Mathematics abounds with such forms and IFI concerns the reading, understanding, and interpreting of such information. [...]

(2) The ability for visual processing (VP). This ability involves visualization and the translation of abstract relationships and nonfigural information into visual terms. It also includes the manipulation and transformation of visual representations and visual imagery.

(Bishop, 1983, p. 184)

Some familiarization and experience (at the exploratory level, but not necessarily at the technical level!) with various types of graphical representations of three-dimensional shapes and relations is a necessary condition for the development of IFI and most probably for the development of some aspects of VP.

4. *The Need for Diversity of Such Representations*

With all due respect to many art educators and to psychologists who stick to the Piagetian tradition for explaining the genesis of the representation of space by individuals, we wish to strongly support the point argued by Josiane Caron-Pargue (1979, ch. 1) that *perspective drawing is just one among many modes of graphical representation, each one having its characteristics and its merits*. The results obtained by L. Páez Sanchez (1980) with adults having little formal schooling as well as some conclusions of the work of Caron-Pargue let us hypothesize the development of 'simultaneous, but inequally accessible' abilities corresponding to different modes of graphical representation: perspective or isometric drawings, coded orthogonal views (like in cartography or in technical drawing), representations by means of layers or sections, ect. Obviously our hypothesis implies the importance of exploratory activities of encoding and decoding of spatial information by means of various types of representations.

EXAMPLES OF ACTIVITIES FOR SCHOOL USE

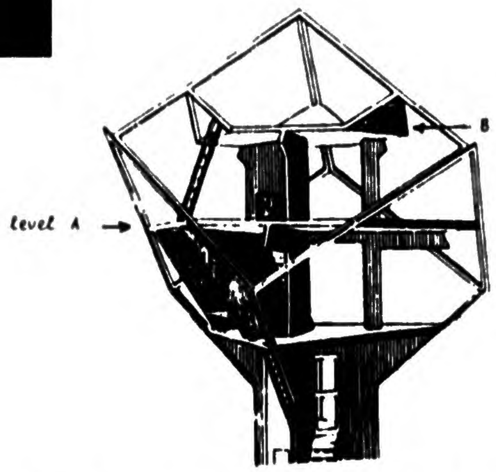
We are now going to examine a few selected examples of activities for school use, taken from existing projects or publications: IOWO (Holland), DIME PROJECT (Scotland), SCHOOL MATHEMATICS PROJECT (Great Britain), MIDDLE GRADES MATHEMATICS PROJECT (U.S.A.), etc. A few guidelines for action will be suggested, with emphasis on what could be done at the middle school level.

>123. Cubehouses do exist. In different towns in Holland (Rotterdam, Helmond) you can find them.
Some inmates of these houses suffer from balance-problems, but otherwise these houses are very comfortable and cozy.

a. Draw the top-view of such a house.
b. As you can see: there are three floors.
   Draw the top-view of the floor of the cube at the A-level.

c. The area of this floor is 60 m^2.
   How long is the edge of the cube?

Cubehouses
architect P. Blom)

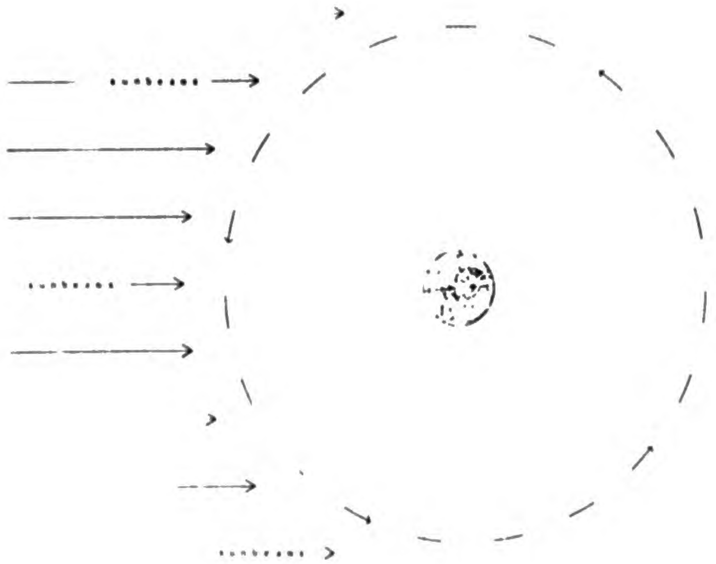


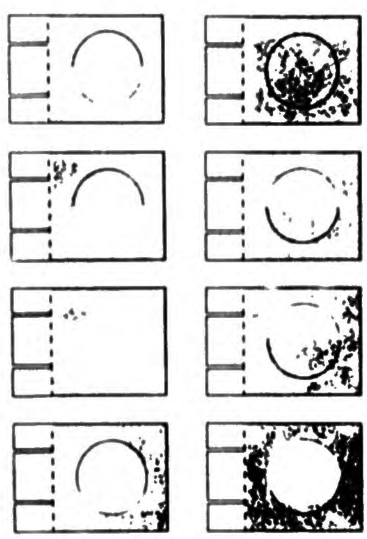
d. How does the area of this floor change when you move the floor upwards a bit (downwards a bit)?
e. The floor of the top-level (B) has the shape of a hexagon. The proof is found in the picture below.
What would the shape of this floor be if it reached to the roof?

f. The height of the living-room is 2.40 m.
What is the height of the attic?

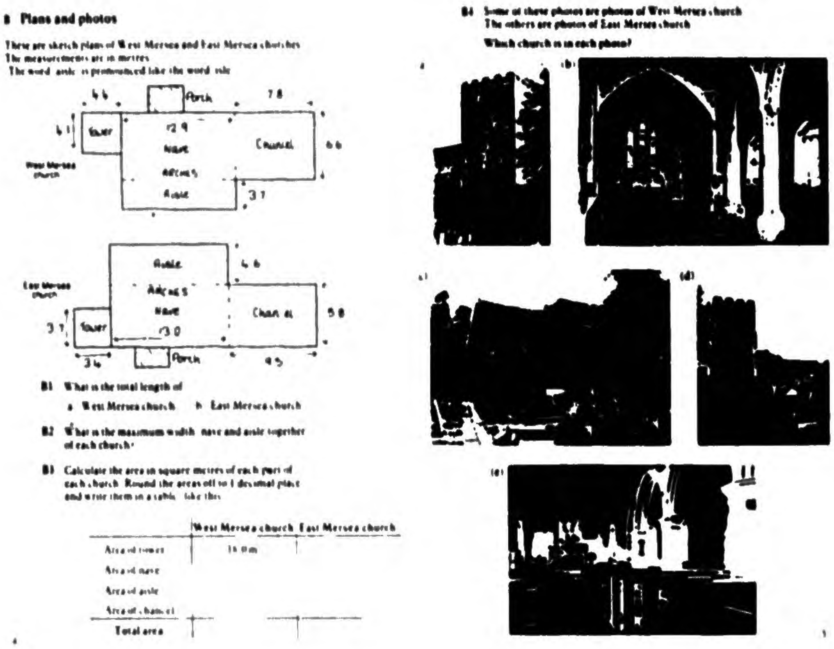
„Activities from Lessons in Space Geometry" (OW & OC). Utrecht, 1982.

▸59. On the following page you see the earth from above. The short arrows indicate the journey which the moon covers every 29 days. Cut out the eight faces of the moon from the next page and fold them as shown in the drawing. Pick a spot in the moon's journey and place there the correct moon-face as seen from the earth. Do this carefully for all eight moon-faces. Show it to the teacher and then glue them in place.





"Activities from Shadow and Depth" (OW & OC). Utrecht, 1980.



B Plans and photos

These are sketch plans of West Mersea and East Mersea churches
The measurements are in metres
The word aisle is pronounced like the word isle

B1 What is the total length of
a West Mersea church b East Mersea church

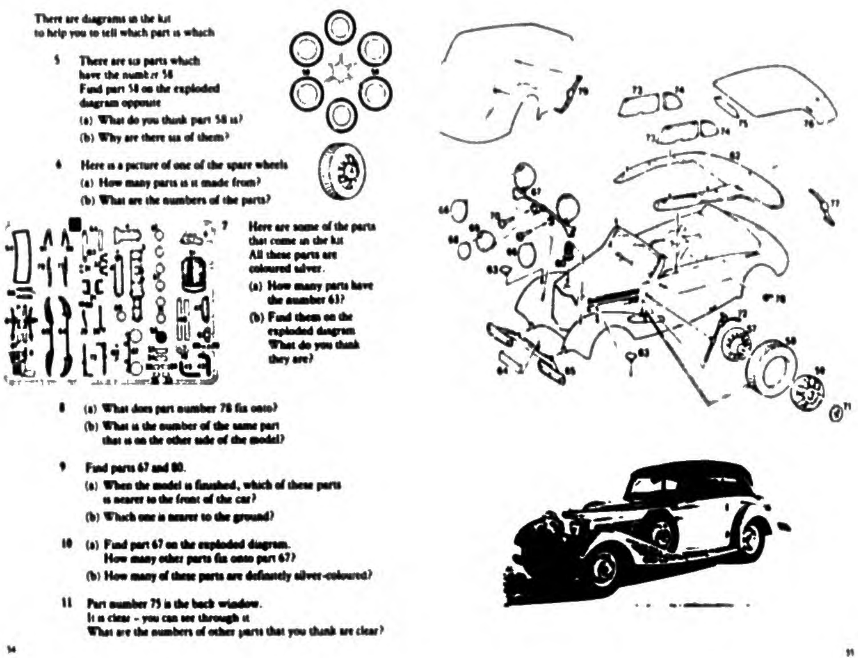
B2 What is the maximum width (nave and aisle together) of each church?

B3 Calculate the area in square metres of each part of each church. Round the areas off to 1 decimal place and write them in a table like this

	West Mersea church	East Mersea church
Area of tower	16.8 m	
Area of nave		
Area of aisle		
Area of chancel		
Total area		

B4 Some of these photos are photos of West Mersea church
The others are photos of East Mersea church
Which church is in each photo?

School Mathematics Project, *Book B2*. Cambridge University Press, 1985. (Reproduced with permission of the publisher.)



There are diagrams in the kit
to help you to tell which part is which

5 There are six parts which have the number 58
Find part 58 on the exploded diagram opposite
(a) What do you think part 58 is?
(b) Why are there six of them?

6 Here is a picture of one of the spare wheels
(a) How many parts is it made from?
(b) What are the numbers of the parts?

7 Here are some of the parts that come in the kit
All these parts are coloured silver.
(a) How many parts have the number 63?
(b) Find them on the exploded diagram
What do you think they are?

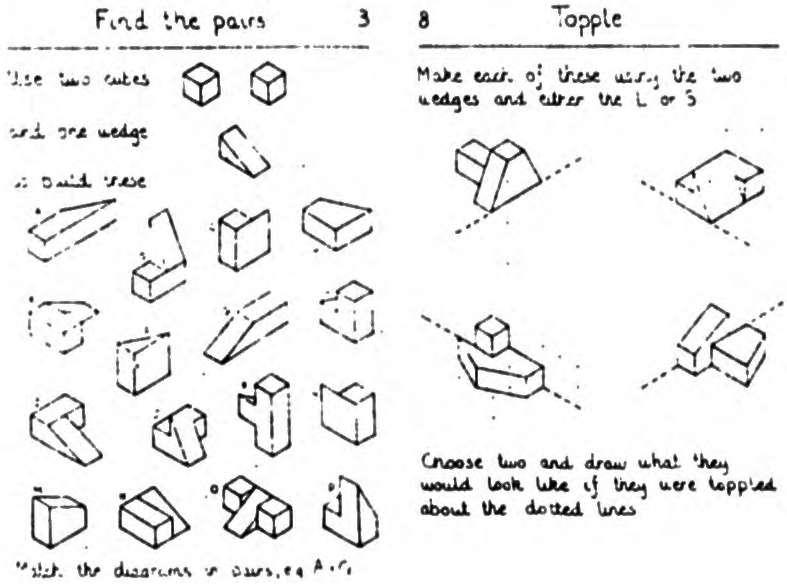
8 (a) What does part number 78 fix onto?
(b) What is the number of the same part that is on the other side of the model?

9 Find parts 67 and 80.
(a) When the model is finished, which of these parts is nearer to the front of the car?
(b) Which one is nearer to the ground?

10 (a) Find part 67 on the exploded diagram.
How many other parts fix onto part 67?
(b) How many of these parts are definitely silver-coloured?

11 Part number 75 is the back window.
It is clear – you can see through it
What are the numbers of other parts that you think are clear?

School Mathematics Project, **Book G6.** Cambridge University Press, 1985. (Reproduced with the permission of the publisher.)



G. Giles, *3-D Sketching Series.* DIME Projects, University of Stirling/Oliver & Boyd, Scotland.

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RICHARD LESH

CONCEPTUAL ANALYSES OF MATHEMATICAL IDEAS AND PROBLEM SOLVING PROCESSES*

I have been asked to address the topic, *conceptual* analyses of mathematical ideas and problem solving processes, and to give examples from: (1) my recent project on Applied Mathematical Problem Solving (AMPS), (2) research with Merlyn Behr and Tom Post on our Rational Number (RN) and Proportional Reasoning (PR) projects, and (3) current computer-based instruction projects at the World Institute for Computer Assisted Teaching (WICAT).

The topic may be particularly interesting to PME members because conceptual analysis techniques are common in the research of members of this group, whereas such a theoretical orientation seldom characterize the work of psychologists who lack mathematical expertise, or mathematics educators who lack psychological expertise.

CONCEPTUAL ANALYSES

What is a conceptual analysis? I will begin by describing three related analysis types: task analyses, idea analyses, and analyses of student's cognitive characteristics.

I will claim that: (a) idea analyses represent a major subset of a more general class, conceptual analyses, (b) conceptual analyses require and emphasize the kinds of knowledge that mathematics educators uniquely are able to contribute to research on learning and problem solving, and (c) currently, some of the most promising research relevant to mathematical learning and problem solving is moving in the direction of conceptual analyses from either a task analysis or analysis of students perspective.

Essentially, deciding to use a task analysis, an idea analysis, or an analysis of students' cognitive characteristics means choosing a unit of analysis that will be 'most useful' for the kinds of practical or theoretical decisions one is trying to make: it is a choice of "how far down to crank the microscope" when examining mathematical learning or problem solving performance.

(a) Analyses of students' cognitive characteristics lead to generaliza-

tions about *students*. Consequently, they tend to generate labels (e.g., concrete operational, impulsive, field dependent, spatial/geometric thinker, gifted, etc.) which are: (i) associated with a given *student*, (ii) assumed to be invariant across content topics, tasks, and time, and (iii) assumed to be difficult or impossible to influence through instruction.

(2) Idea analyses lead to generalizations about various conceptualizations of particular *ideas*. They focus on cognitive processes and capabilities that are linked to specific content understandings, and *variability* across content and context is explained in terms of *conceptual* understandings. It also is assumed that learning and problem solving performance characteristics can be modified through instruction, and that content-*dependent* learning and problem solving processes are more important than general heuristics.

(c) Task analyses go one step further than idea analyses concerning the issue of variability. Compared with idea analyses, little attempt is made to explain similarities in behavior across tasks characterized by the same idea. Compared with analyses of students' characteristics, task analyses assume that variability within an individual across tasks is more important than variability within tasks across individuals (Hayes and Simon, 1977; Newell and Simon, 1972). Task variables are of prime interest (Anderson *et al.*, 1981; Kulm, 1979; Golding and McClintock, 1979; Hayes, 1981; Simon, 1979).

CONCEPTUAL ANALYSES OF PROBLEM SOLVING, STRATEGIES, AND HEURISTICS

In the past decade, some of the most productive areas of mathematics education research have aimed at clarifying the nature of students' primitive conceptualizations of a variety of mathematical ideas, e.g., early number concepts, rational number concepts, or spatial/geometric concepts.

THE ACQUISITION AND USE OF MATHEMATICAL CONCEPTS AND PROCESSES

Lesh and Landau (1983) includes chapters by many PME members: e.g., Behr, Bishop, Carpenter, Moser, Post, Vergnaud. Recently, some of these idea analyses have begun to evolve into studies of problem solving processes, heuristics, and student characteristics (e.g., Carpenter *et al.*, 1982; Lesh, note 1; Lester, 1983; Schoenfeld, 1985).

The evolution has taken two forms:

(1) While investigating the evolution of a basic mathematical concept, interdependencies between content-understanding and process-use emerge as important. For example, many 'modeling' and representational processes contribute both the underlying *meaning* of basic mathematical ideas, as well as to their *usability*.

Freudenthal's emphasis on 'teaching mathematics so as to be useful' reflects this approach, as does the work of Behr, Post, and myself in the area of proportional reasoning and rational number concepts. Freudenthal's perspective is very different from those who support 'teaching problem solving' or even 'applied problem solving'.

(2) Techniques that have been used effectively to investigate what it means to 'understand' given *ideas* are applied directly to problem solving processes, heuristics, and understandings.

The Applied Mathematical Problem Solving (AMPS) project has used this second approach. In the AMPS project, we assume that as students' mathematical *concepts* are developing, the processes, heuristics, and strategies associated with these concepts also are developing. Consequently, we investigate primitive conceptions of problem solving processes and heuristics using 'idea analysis' techniques similar to those we have used to investigate primitive conceptualizations of rational number or proportional reasoning concepts (Behr *et al.*, 1983) or spatial/geometric concepts (Lesh and Mierkiewicz, 1978).

Schoenfeld (1985), Lester (1983), and Silver (1985) are three other mathematics educators whose research tends to use this general approach. Schoenfeld's metacognitive understandings (1983a), managerial functions (1983b), and belief systems (1983a) are components of heuristic understanding. Or, the research of Landau (1983), Lester (1983), and others has made it clear that 'understanding' a given heuristic or process, like 'draw a picture', means: (i) knowing *when* it should be used, (ii) knowing *how* it is related to other heuristics, (iii) knowing *which* of a variety of pictures best fit a particular idea, situation, or set of relationships, and (iv) knowing *that* the heuristic may serve different functions at different stages in a problem's solution (e.g., interpretation stages versus verification stages, etc.).

One of the first assumptions that an 'idea analysis' perspective tends to impart on problem solving research is that *ideas develop*; they do not go from 'not known' to 'understood' in a single step. This idea-developmental assumption has important and far-reaching implications for problem solving research. For example, in the AMPS project, when

our students used mathematical ideas to solve problems, their ideas tended to be at intermediate stages of development, their conceptualizations of the underlying ideas(s) (or sets of ideas) had to be refined and adapted to fit the problem situation; the underlying conceptualizations actually *developed* (locally) during forty minute problem solving episodes. Consequently, the problem solving mechanisms (e.g., processes, skills, and understandings) that were helpful were those that facilitated local idea development. Conversely, many heuristics that other researchers have reported to perform productive functions in the presence of mature understandings actually tended to be counter-productive in the presence of primitive conceptualizations.

Later in this paper, more will be said about these and other implications of an idea-development perspective on problem solving research and instruction. First, it is useful to identify several more important similarities and differences between *child development* research and *idea development* research.

IDEA ANALYSES VERSUS ANALYSES OF STUDENTS' COGNITIVE CHARACTERISTICS

Because human development so often is described in terms of conceptual capabilities related to particular logical/mathematical ideas, idea development research frequently appears to be indistinguishable from child development research. The differences may be subtile, but they are important.

Human development research results in generalizations about *students* rather than about students' *ideas*. Idea development research results in generalizations about behaviors that can be expected from a student who has acquired a particular conceptualization of given ideas (or set of ideas).

Research on individual differences or cognitive style illustrates the distinction between idea analyses and analyses of students' cognitive characteristics. 'Individual difference' research is founded on the observations that different students respond differently to identical problems or stimuli. The additional assumption also usually is made, however, that useful (?!!!) characteristics must be invariant across large content domains, and across diverse sets of tasks and situations. The emphasis is on demonstrating and explaining *similarities* in behavior across seemingly unrelated situations; variability across tasks has been of less interest. Instructional goals consist largely of finding ways to assign labels

to students which will allow them to be sorted into groups, perhaps for the purpose of providing differentiated educational experiences (Gould, 1981). Idea analyses, on the other hand, assume that the students' tendencies can be influenced through instruction; it is not necessary simply to compensate for characteristic 'givens'.

When comparisons are made between 'gifted' students and 'average ability' students, or between 'good problem solvers' and 'average problem solvers', an 'analysis of students' cognitive characteristics' perspective implicitly tends to be taken; the content-independent nature of processes, heuristics, and characteristics tends to be emphasized.

'Idea analyses' assume that 'giftedness' varies across subject matter domains, and that a problem solver who is 'good' at problems related to one topic may be 'not so good' at others. Idea analyses anticipate that most productive heuristics and strategies will be content-*dependent*. For example, in a number of recent research studies focusing on problems involving the application of substantive content understandings, the utility of all-purpose, content-independent processes has been challenged (e.g., Elstein *et al.*, 1978; Rogoff and Lave, 1983; Lave, 1984). Research in conceptually rich problem solving situations has suggested that it often is *poor* problem solvers in the relevant content domain who use general-but-weak strategies such as 'working backwards', 'hill climbing', or other 'means-ends analysis' techniques; good problem solvers in the domain tend to use powerful content-*related* processes (e.g. Larkin *et al.*, 1978).

Results from our AMPS project yield similar conclusions: (a) Students who *have* substantive ideas to bring to bear on a problem tend to *use them*, together with powerful content-related processes; (b) Students who do not have relevant ideas in a particular domain are, in general, poor problem solvers in that domain, even if they *have* had extensive training in the use of general, content-independent heuristics and strategies (Lesh, note 1).

Sometimes problem solving research assumes content-independence in subtle ways. For example, heuristics often are assumed to provide answers to the question, "What should I do when I am stuck?" This question transforms easily into, "What should I do when I have no substantive ideas to bring to bare on the problem?" Research relevant to this later issue can focus on problems in which no substantive ideas (and consequently no content-related processes) are needed.

The results, however, may have little relevance to conceptually rich problem solving situations.

In our AMPS project, because of the nature of the problems and subjects that we investigate, we have seen virtually no conscious or overt uses of commonly discussed heuristics. We *have* however seen many instances of behaviors which appear to be based on primitive versions of heuristics and strategies. At this stage of our research, it is by no means clear that these behaviors reflect first steps in the direction of mature strategic or heuristic understandings. Furthermore, even if developmental links can be traced, it is not clear that instruction should parallel development. However, it does seem sensible that instructional considerations should be informed by developmental knowledge. As a later section of this paper will explain, many heuristics and strategies appear to require significant reconceptualization to prevent them from yielding counterproductive results when they are used by students with unstable conceptual systems (a term I will discuss later in this paper). In the same way that idea analysis research has shown that it is naive to speak of students 'having' or 'not having' particular ideas, it seems reasonable to assume that problem solving processes, heuristics, and characteristics should be submitted to developmentally-oriented conceptual analyses. What is the nature of students' primitive conceptualizations of particular processes or heuristics, and what factors hinder or facilitate their evolution?

NATURAL DEVELOPMENT VERSUS DEVELOPMENT IN MATHEMATICALLY RICH ENVIRONMENTS

Another distinctive characteristic of human development research is that it tends to focus on very general ideas that develop '*naturally*'; whereas idea development research often focus on ideas that are unlikely to evolve outside artificial (perhaps instructional) settings.

Whereas naturalistic observations can be used to trace the development of ideas that evolve naturally, many mathematical ideas are unlikely to evolve outside of mathematically rich instructional environments. Furthermore, mathematics educators tend to be less interested in what students can do 'naturally', than with what they can do using 'conceptual amplifiers' like powerful and economical representational systems, or language and symbol systems (Vygotski, 1978). For these reasons, idea development research in mathematics education often must rely on interventionist 'teaching experiments' or longitudinal development studies in artificial but mathematically rich environments. A major goal of idea development research is to describe the impact of 'capability amplifiers' on the development and use of particular ideas. The focus is on *ideas* and

the role of powerful and economical *amplifiers*, not necessarily on 'natural' development or behavior.

These emphases are becoming even more important due to the availability of powerful new technological tools. For example, with computer-driven utilities, like WICAT's algebraic 'symbol manipulator, function plotter' or geometric 'proof checker, graphics editor' (or even familiar tools like VisiCalc), problem solving in the presence of such 'conceptual amplifiers' is becoming as important as that in their absence. In fact, for mathematically talented students who should be encouraged to pursue careers in mathematics, science, and engineering, these emphases are even more significant. The learner or problem solver no longer can be assumed to be a student working alone with only a pencil and paper for tools.

From the point of view of theory development, assumptions that are relevant to an 'amplified organism' may need to be considerably different from those common in past cognitive science studies.

From the point of view of instructional development, traditional wisdom also may need to be revised. For example, using powerful new computer utilities, realistic applications can be used to *introduce* a wide variety of mathematical topics. Then, we can gradually guide students to build their own conceptualizations of the underlying idea (Fey, 1984).

By minimizing the tediousness of answer-giving procedures, we can focus a student's attention on non-answer-giving phases of problem solving, where activities like information filtering, data gathering and organizing, problem formulation, and trial solution evaluation are involved. By focusing attention on the underlying conceptualizations of problem situations, and on the sensibility of products of thought, subtilties about the meanings of the underlying ideas become apparent. Also, by reducing the conceptual energies devoted to 'first order thinking', higher order 'thinking about thinking' becomes possible. Otherwise, students frequently become so embroiled in 'doing' a problem that they are unable to think about *what* they are doing, and *why*. Using utilities, on the other hand, we can provide students with 'ledger sheets' of their solution paths, so they can explicitly examine and modify and refine their thinking processes.[1]

At WICAT, we have compared groups of students receiving a traditional 'teach first, apply later' instructional approach to groups receiving a computer-utilities-based 'applications first, conceptualization refinement' approach. The results have shown that, for comparable amounts of instructional time-on-task, the utilities groups consistently have

outperformed the control groups; and they have done so not on 'applications' or 'problem solving' post-test items, and on items focusing on the meanings of the underlying concepts, but also on 'computation' items associated with the topics – even though the utilities groups had no practice at all on the pencil-and-paper computations.

IDEA ANALYSES VERSUS TASK ANALYSES

Recently, some of the most theoretically interesting types of task analyses have been accompanied by attempts to create artificial-intelligence (AI) based information-processing (IP) models to simulate students' problem solving performance (Briars, 1982; Greeno, 1980; Heller and Greeno, 1979; Mayer, 1983). Compared with strict task analyses, which are intentionally naive with respect to internal processes, IP task analyses are midway between idea analyses and task analyses. AI-based IP models assume that students' interpretations of a given task are influenced as much by internal 'programs' or representations as by external task variables. Still, it is one thing to create a program which behaves similarly to humans on a given task, and quite another to create a program simulating the way students' substantive *ideas* influence behavior on the task. Sufficiency in the former sense is considerably different from sufficiency in the latter. Teaching a student to perform a task is not at all the same as introducing the student to a conept which can be applied to an unbounded cluster of tasks (Greeno, 1983), and which is related in non-arbitrary and substantive ways to other concepts (Ausubel, 1963; Lesh, 1976).

Both idea analyses and AI-based task analyses assume that the organizational/relational systems that mathematicians use to interpret a task may not correspond to the ones used by youngsters (Hayes and Simon, 1977). This is why task analyses begin with a set of tasks, and *then* demonstrate task relatedness based on detailed observations of student behaviors; that is, they begin with a set of tasks, and *then* create a model (note: the model is the researcher's) to explain the student's capabilities and understandings by simulating behaviors. In contrast, idea analysis begin with models (i.e., hypotheses about the structures that characterize students' mathematical ideas), and *then* create tasks to test these hypotheses.

In our RN and AMPS projects, theoretical perspectives have been influenced strongly by biases about the nature of mathematics and what it means to *do* mathematics. The following two hypotheses are central

to our perspective, and they tend to be distinguishing characteristics of a 'conceptual analysis' approach to problem solving.

(i) In real situations, for students to make judgements involving mathematical ideas, they must 'read in' *some* organizational/relational system in order to 'read out' mathematics-relevant information. Mathematics isn't *in* things, it is the study of structures that are *imposed on* things; that is, the content of mathematics consists of structures, and to do mathematics is to create and manipulate structures. These structures, whether they are embedded in pictures, spoken language, real objects, or written symbols, are skeletons of the 'conceptual models' that mathematicians and mathematics students use to interpret and solve problems.

A major goal of our RN projects has been to describe in detail the nature of students' primitive conceptualizations of a series of central rational numbers and/or proportional reasoning ideas. We also attempt to describe the evolution of the underlying "conceptual models" associated with these ideas.

(ii) Our second hypothesis is that many of the most important factors influencing learning and problem solving capabilities are directly related to the stability (e.g., wholeness or degree of coordination, internal consistency, consistency with the modeled world) of the relevant conceptual models. In particular, this view assumes that one can investigate mechanisms influencing the evolution of conceptual models without necessarily knowing all of the details about the exact structural characteristics of the underlying idea(s). This has been the general approach adopted by our AMPS research.

In both our RN and AMPS projects, we are not simply interested in describing 'states' of knowledge, we are interested in the way transitions are made from one state to another. Our goal is to model students' modeling behaviors (Lesh, 1983). Like AI-based IP models of problem solving, we consider the student to be an adaptive organism whose interpretations of problems are influenced by internal models as well as by external stimuli. However, our student is a 'modeler' more than a 'processor', and mathematics furnishes the 'conceptual models' for interpreting and transforming problem situations.

In the past, most mathematics-relevant AI-based IP models had only attempted to simulate various *states* of knowledge. More recently, some models are attempting to describe the way transitions are made from one state to another (Chi, 1978; Klahr and Wallace, 1976; Larkin, 1982; Resnick, 1983; Riley *et al.*, 1983; Siegler, 1981, 1984). Still, these descrip-

tions tend to treat cognitive growth as incremental and quantitative, i.e., as process of adding and deleting specific productions. This restricted view of cognitive adaptation is not an inherent property of AI systems; rather, it appears to result from the tendency of certain types of AI models to represent mathematical ideas as node-like entities with little or no internal structure (Newell, 1972).

In mathematics learning and problem solving research, studies which de-emphasize the internal complexity and 'holistic' character of conceptual structures tend to hypothesize relatively powerful processes; whereas studies hypothesizing the existence of *powerful* relational/organizational structures need only relatively weak processes.

When research is based on a 'weak structure, powerful process' perspective, cognitive growth tends to be described in terms of incremental and cumulative changes; (quantitative) additions and deletions of procedures are emphasized rather than (qualitative) reorganizations of structured wholes. A 'powerful structure, weak process' perspective, on the other hand, predisposes researchers to confront and explain insights, intuitions, and other conceptual discontinuities (Fischbein, 1983).

In both our AMPS and RN research, the development of conceptual models has been characterized by *both* incremental quantitative growth *and* by discrete qualitative discontinuities. Models of cognition which encounter difficulties reconciling continuous changes with discrete jumps often do so because mathematical concept development and problem solving are described using models which do not allow a whole system to be more than the sum of its parts – as though psychological units (analogous to chemical units of, say, hydrogen and oxygen) can never be combined to form something with new properties (like water), and as though continuous or incremental quantitative changes imposed on the system can never give rise to qualitative structural changes (as when changes in temperature turn water into ice or steam).

Mathematics is a domain of knowledge in which form and content not only are inter-related, to a large extents, conent *is* form. Many ideas are characterized by structures which are virtually *non*-decomposable; that is, the mathematical meaning of the parts are derived *entirely* from the system in which they are elements. In past articles, I have discussed this phenomenon using the rubric of 'structual integ ation'; Piaget (Piaget and Beth, 1966) refers to the phenomenon as 'reflective abstraction'. Relevant analogies from non-mathematics disciplines are less like hydrogen and oxygen molecules in water (which is perhaps no more than 'a nearly decomposable system) than they are like 'renormalized particles'

or quarks in quantum physics. Inside a proton or neutron, quarks not only cannot be seen as clearly identifiable parts, they cannot even be isolated and pulled out. Virtually *all* of their 'meaning' derives from the systems in which they are 'embedded'.

A decade ago, Newell (1973) stated, "Our task in psychology is first to discover that structure which is fixed and invariant (underlying a set of tasks) so that we can theoretically infer the method (used to perform the task) – Without such a framework within which to work, the generation of new explanations will go on ad nauseum. There will be no discipline for it, as there is none now." (p. 296). Today current research at the interface of mathematics and psychology is only beginning to address Newell's challenge.

Clearly, some of the best AI-based IP models are beginning to take into account some important structural properties of mathematical ideas, and some of the best conceptually-focused mathematics education research is beginning to achieve a degree of specificity comparable to that which once was attainable only in the context of extremely restricted task domain in these trends continue, both vague idea analyses and naive task analysis move progressively closer toward a form of *conceptual* analysis.

FINAL THEORETICAL CONSIDERATIONS

Unlike a great deal of the best current problem solving research in substantive content domains, the theoretical descriptions and mathematical models we use do not fit the characterization of 'information processing systems'. Our explanations of problem solving and concept formation tend to be more 'organismic' than 'mechanistic'. Our theoretical constructs bear closer resemblances to many of Piaget's ideas than to artificial intelligence models. We *do* treat the learner as an adaptive system whose interpretation of problems is influenced by internal models, as well as by external stimuli, but we do not treat mathematics as information to be processed, nor mathematicians as processors. For us, the mathematician or mathematics student is considered to be a 'situation interpreter and transformer', and mathematics furnishes the 'conceptual models' for making interpretations and transformations.

If information processing was going on in our sessions, the information was not simply raw data, or even multiple attributes associated with raw data; it was the imposed organizational/relational systems that were being transformed. That is, it was primarily the conceptual models themselves that were being 'processed' during solution attempts (Lesh, 1983).

During solution attempts, the most beneficial decisions were seldom procedural; "What shall I *do* next?" Students who seemed preoccupied with *doing* typically did not *do* well compared with their peers. Beneficial considerations tended to be *conceptual* in nature, focusing on thinking about ways to *think about* the situation (i.e., relationships among 'givens', or interpretations of 'givens' or 'goals'), rather than about ways to *do it*, or get from 'givens' to 'goals'. This 'conceptual versus procedural' distinction was especially important during early stages of solution attempts when students' conceptual models were most unstable (Lesh and Zawojewski, 1983).

The most appropriate general characterization of most of our AMPS problem solving sessions is that our students constructed solutions by gradually organizing, integrating, and differentiating unstable *conceptual* structures (e.g., conceptual models) more than by linking together stable *procedural* systems (e.g., production systems associated with AI-based IP theories).

If problem solving is characterized as a process of linking together stable *procedural* systems, then the most important heuristics and strategies may appear to be those which help the problem solver select and sequence their activities. That is, they serve 'managerial' or 'executive' functions governing the use of more powerful, less general, and more content-dependent processes. However, if (as in most of our AMPS sessions) solutions are constructed by gradually refining, integrating, and differentiating unstable *conceptual* systems (i.e., conceptual models), then the most important heuristics and strategies are those dealing with: (a) how deficiencies in various models are detected, (b) how to minimize the debilitating influences associated with the use of unstable conceptual models, (c) how successively more complex and refined models are gradually constructed, and (d) how competing interpretations are differentiated, reconciled, and/or integrated our sessions, many of the most effective activities facilitating solution attempts functioned not so much to help the problem solver amplify his/her problem solving abilities as they did to help the problem solver minimize cognitive characteristics associated with the use of unstable conceptual models.

Several years ago, when 'Piagetians' attempted to specify implications for classroom practice, the question, "Can cognitive development be accelerated?" was referred to as the 'American question'. Our AMPS research is showing that Piaget's theory, as well as several other 'developmental' perspectives (e.g., Vygotskii in psychology, Gould in biology, Lakatos (1963) in the history and philosophy of science) are ap-

pearing to have new relevance. The new relevance results from the nature of the problems that we address, and because of the 'local conceptual development' character of typical solution attempts. The new 'American question', appears to be "How can we minimize cognitive deterioration (or conceptual difficulties that are naturally associated with unstable problem conceptualizations)?"

In our problem solving sessions, we frequently found students doing poorly because of things that they *had learned* in their mathematics classes about the nature of mathematics and mathematical problem solving. Or, activities (heuristic, strategies, processes, etc.) which were beneficial at one stage of problem solving were counter-productive at others. Beneficial activities varied across problems, and across stages in the solution of individual problems. Our 'conceptual analyses' of students' modeling will continue to explore parallels between:

(a) Mechanisms that influence in our 'local conceptual development' sessions and mechanisms that facilitate or hinder general (or 'natural') cognitive development. For example, Piaget (1971) and Vygotskii (1978) have different explanations for roles that peer group interactions can play to help students acquire certain problem solving skills.

(b) The 'local' evolution of adaptive *conceptual* systems and the evolution of biological organisms (e.g., Gould, 1980; Piaget, 1971). For example, biologists have described some of the most important mechanisms that propel evolution, and they have described circumstances in which evolution is not likely to lead to 'higher order' organisms. The popular picture of evolution as a continuous sequence of ancestors and decendents is misleading. Biological evolutionists in general see periods of rapid change followed by long periods of tranquility; several lineages frequently coexist, none clearly derived from another. Gould states:

> The 'sudden' appearance of species – is the proper prediction of evolutionary theory as we understand it. Evolution usually proceeds by (local) 'speciation' – the splitting of one lineage from a parental stock – not by the slow and steady transformation of the large (central) parental stock. Repeated episodes of speciation produce a bush. Evolutionary 'sequences' are not like rungs on a ladder (Gould, 1980, p. 61).

(c) 'Idea development' in the context of particular problem solving situations and 'knowledge development' as it is interpreted in recent 'history of science' perspectives (Gould, 1980; Kuhn, 1971; Lakatos, 1970; Suppes, 1977 e.g.). Early development stages frequently are characterized by the coexistence of a number of distinct (in retrospect)

yet relatively undifferentiated (at that time) world views. Development-by-accumulation often fails to explain important aspects of conceptual evolution; conceptual reorganizations often radically alter perceptions of what is 'intuitive' or 'obvious', etc.

Each of the above three suggestions are ripe with notions that run counter to traditional theoretical perspectives in mathematics learning and problem solving, but which are very consistent with the 'local conceptual development' character of our AMPS sessions.

CONCEPTUAL ANALYSES OF MATHEMATICAL PROBLEM SOLVING: AN EXAMPLE

The AMPS project differed from most problem solving projects in mathematics education because it evolved out of 'idea analyses' aimed at clarify the nature of youngsters' primitive conceptualizations of rational number concepts, spatial/geometric concepts or early number or measurement concepts. It *has not been* the goal of the AMPS project to study problem solving per se; rather our goal is to enrich what educators mean when they say a student *understands* a mathematical idea. We have investigated the processes, skills, and understandings that average ability students need in order to adapt their mathematical ideas to model everyday situations. Our ultimate instructional objective is to help average ability students use ideas that they *do have*, not to function better in situations in which they have none, or in puzzle or game-like situations in which no substantive ideas are relevant.

A surprisingly small percentage of the problem solving research is more that tangentially relevant to questions about using mathematics in realistic settings. Burkhardt's research is a notable exception (1983).

AMPS research has shown that seemingly realistic 'word problems' often differ significantly from their real-world counterparts with respect to difficulty, processes most often needed in solutions, and error types most frequently committed (Lesh *et al.*, 1983). Furthermore, if one identifies salient characteristics of everyday situations in which mathematics is used, many of the most important problem types are not represented at all in most textbooks (Bell, 1979).

At the 1978 PME meeting in Osnabrück, I argued that research on mathematical problem solving would be significantly different if the following were not neglected: (a) real mathematics, (b) realistic situations, and (c) real (i.e. average ability) students. These rationale, in fact, led to the AMPS project. Consequently, after six years of study, it seems

appropriate that I re-examine my claims and modify or extend them in the light of AMPS results.

(1) *Focus on Real Mathematics*

The problems we have investigated involve easy-to-identify elementary mathematics concepts (i.e., basic arithmetic, measurement, and number ideas). Consequently, like others whose research has focused on problems involving the application of substantive content understandings, we have found that successful problem solvers tend to use powerful content-*dependent* processes rather than general content-*independent* processes (e.g., Simon, 1981; Elstein *et al.*, 1978). The processes that we have found to be most beneficial to our students contribute to both the *meaningfulness* as well as the *usability* of the relevant mathematical concepts. They are integral parts of the underlying conceptual models.

In our problems, alternative ways of conceptualizing problem situations required different organizational/relational systems ot be *imposed on* the problem solving situations, and these distinct systems resulted in different ways of filtering, organizing, and interpreting 'givens' and 'goals'. Students' initial problem conceptualizations tended to be barren and distorted compared with final conceptualizations, and 'stages' in the development of a given conceptualization could be identified depending on the complexity and refinement of the underlying systems.

The preceding kinds of observations led us to recognize one of the most fascinating and potentially powerful conclusions of our research; that is, many of our forty-minute sessions resembled compact versions of phenomena that developmental psychologists have observed over time periods of several years (e.g., concerning the evolution of particular mathematical ideas).

We have characterized our problem solving sessions as 'local conceptual development' episodes. For example, in our 'inflation' problem (Lesh, 1983), a problem that involves proportional-reasoning, the sequence of stages that our students went through were remarkably similar to accounts of the 'natural' development of proportional-reasoning capabilities given by Piaget (1966), Noelting (1979), Karplus *et al.* (1983) and others. The students moved from conceptualizations that focused on only the most salient or superficial characteristics, and that dealt with facts (or relationships, or information) one at a time, to dealing with *sets* of variables, several at a time, and in well organized systems.

Primitive versions of given conceptualizations (i.e., conceptual models) were based on relatively unstable (e.g., poorly coordinated) rela-

tional/organizational systems; and these poorly coordinated systems tended to:

(a) notice only the most salient relationships and information in the problem situation, filtering out other important but less striking characteristics; and/or

(b) neglect to notice model-reality mismatches, thus imposing subjective and unwarrented relationships or interpretations based on 'a priori' assumptions. Progressively more stable versions of given conceptualizations took into account more information (and more different kinds of information), and to impose fewer irrelevant or debilitating restrictions, or subjective false assumptions. An initial interpretation (i.e., model) selected, organized, and interpreted a subset of the available information; this interpreted information then required certain aspects of the model to be refined or elaborated; and the refined or elaborated model allowed new information and relationships to be noticed, thus giving rise to a new 'modeling' cycle.

Conclusions Related to Emphasizing Substantive Mathematical Ideas

Although AMPS transcripts illustrate many mathematical skills and understanding that are distressingly deficient in most students, our students did prove to be fairly able modelers if they are given the opportunity to exhibit these capabilities. They routinely 'invented' new ideas, or significant extensions of familiar ideas. Furthermore, it was easy to identify processes, skills, and understandings that (a) are not taught in school, (b) are important sources of difficulty as students try to use their mathematical ideas in everyday situations, (c) are not difficult to teach, (d) would make students significantly better 'real world' problem solvers, and (e) would fit Freudenthal's emphasis on teaching *mathematics* – but teaching it so as to be useful.[2]

(2) *Focus on Realistic Situations*

AMPS problems are simulations of realistic problem solving situations that might reasonably occur in the everyday lives of our students or their families. They have dealt with shopping, sports teams, newspaper information, planning a summer job or party or trip, etc. Many of these problems and their characteristics have been described elsewhere (e.g., Lesh, 1981; Lesh and Akerstom, 1981; Lesh and Zawojewskii, 1985; Nesher, 1980). Most of our problems required at least 15–40 minutes to complete, and realistic 'tools' were available (e.g., calculators, rulers, graph paper, reference books, etc.). A variety of solutions and solution paths

also were available, so evaluating the usefulness or quality of trial solutions or solution paths was important. The project had shown that a characteristic of good everyday applied problem solvers is their ability, upon confronting a problem, to quickly and accurately assess problem difficulty, needed resources, and time required for an adequate solution. A 30 second solution attempt is very different from a 5 minute attempt or 30 minute attempt.

Our problems seldom could be characterized as situations in which students needed to get from explicit 'givens' to well defined "goals" using clearly specified procedures. The definitions of problems, givens, goals, and acceptable solution paths, all required input from students. Unlike textbook word problems in which either 'too much' or 'not enough' information is given, many of our problems involved an overwhelming amount of information, all of which was relevant, and the main difficulty was to select and organize the information that was 'most useful' in order to find an answer that is 'good enough'; or, not enough information was available, but a useable answer had to be given anyway. In other cases, additional information had to be generated or gathered during the solution process; all of the relevant information was not given at the start.

For most of our problems, non-answer-giving stages (e.g., problem formulation, trial solution evaluation, etc.) were critical. The goal was not simply to arrive at a mathematical answer; mathematics was a means to an end, not an end in itself (Lesh and Akerstrom, 1981), and acceptible solutions varied in both type and sophistication.

All of the above characteristics of realistic situations are related to the use of mathematical ideas as models that select, filter, and 'fill in holes' within information that is given in a problem solving situation. Consequently, investigating the usefulness or adequacy of trial models was an important activity.

We found that the most appropriate characterization of most of our AMPS sessions was that the students alternated among 2–4 distinct interpretations or models of the situation (including goals and givens), with different conceptualizations often recurring at several distinct stages of complexity, refinement, and stability. We have examples of sessions in which students want through as many as ten 'modeling cycles' during a forty minute session (Lesh, 1982). Differences between model types, or between different forms of a given model type, were based on: (i) different data being selected or filtered, (ii) different types of relational/organizational systems being *imposed on* the data, or (iii) dif-

ferent ways of simplifying, combining, or synthesizing the data.

During early stages of many of our sessions, several half-formulated (often logically incompatible) conceptualizations operated simultaneously, each suggesting half-formulated solution procedures and/or alternative ways to select, filter, interpret, relate, organize, or synthesize information. Students often moved from one conceptualization to another in the following general way. First, a particular type of relationship was noticed which was associated with a given conceptualization. Then, as attention shifted away from the overall conceptualization, the relationship began to be treated as part of an entirely different conceptualization. The phenomenon resembled a repeating cycle of 'losing the forest when looking at the trees' and 'losing the trees when looking at the forest'. This sort of 'mutation' from wholes, to parts, to new wholes, to new parts, sometimes led to new productive ideas or interpretations; in other cases, it derailed promising solution efforts.[3]

Conclusions Related to Emphasizing Relastic Situations

Because of the nature of our students and problems, we have had to reconceptualize or redefine many popular heuristics and strategies to make them suitable for: (a) non-answer-giving stages of problem solving, (b) processes that are content-dependent, (c) solution paths involving multiple 'modeling cycles', where the first conceptualizations can be expected to be barren and distorted, and (d) conceptual models based on unstable organizational/relational (as well as procedural) systems.

We have shown that a given heuristic or strategy may have either positive or negative consequences depending on: (i) its content-relatedness, (ii) its conceptual versus procedural focus (a distinction that I will make in a moment), (iii) the function it is to serve at a particular 'stage' in which it is used, and (iv) the stability of the underlying conceptual model. Stage-dependent, content-dependent, conceptually-focused techniques tend to be most beneficial when their main function is to help problem solvers 'minimize the negative influences of unstable conceptual systems' more than to 'maximize the effectiveness of stable procedural systems' (Lesh, 1983).

Our research has made us sensitive to the tendency of educators to turn a 'means to an end' into an 'end in itself'. Heuristics should be taught in a way that does not conceal the goals that give rise to them (Noddings, 1983). For example, in problem solving situations, one does not 'look for a similar problem' as an end in itself; the goal usually is to 'look for a similar problem *in order to better understand the "given" problem*'. One

does not draw a picture as an end in itself; the picture is drawn for some *purpose* – and the most useful purposes facilitate one or more of the *functions* that are critical to 'local conceptual development'. A major goal of our research is to identify some of the most important of these 'local conceptual development' functions or mechanisms. To identify these functions we treat both *substantive mathematical ideas* and problem solving *processes* (or *heuristics*) developmentally.

(3) *Focus on Real Students*

AMPS research has not found it useful to compare high ability (or 'gifted') students with average ability students, nor have we compared 'experts' with 'novices'. In fact, we have found that the strategies and techniques that 'good problem solvers' use to attack problems when they are 'stuck' (i.e., when they have no substantive ideas to bring to bear on the situation) often are counterproductive to average ability students, who are attempting to use ideas that they *do have* but which are based on 'unstable understanding' (a term I will describe below).

The most relevant comparisons for our concerns have been between 'average ability novices who succeed' and 'average ability novices who fail' on a given problem. Or, because the productivity of a student's activities often vary considerably within a 40 minute session, it is even more accurate to say that our comparisons have been between beneficial versus non-beneficial behaviors of average ability students as they attempt to use elementary mathematical concepts in everyday situations.

In our AMPS project, a problem has been defined to be a meaningful situation in which a stable conceptual model *is not* available to the student (or group). Problems, according to our restricted definition, cannot be solved simply by linking together stable systems. Experts, on the other hand, are precisely those people who have acquired stable and accessible conceptual models for interpreting and manipulating information in a given problem domain. Therefore, even when these individuals do not have an immediate response to a question, they *do* have well organized conceptual and procedural systems for producing a response. Thus, the situation is not really a problem at all for these individuals; it is just a (perhaps complex) exercise. Behaviors of 'experts' in a given problem domain therefore can be expected to be qualitatively different from that of novices.

According to our definition, a 'problem' is not characterized as an inability to get from *A* to *B*. So, for example, a mountain climber's 'problem' is not so much to get from the bottom of a cliff to the top, as it

is to 'understand the terrain'. Once the terrain is understood, the activity of getting to the top of the cliff is an exercise, not a problem.

The problem solving behaviors of 'gifted' students also can be expected to be qualitatively different from those of average ability students. For example, gifted students 'see' connections among problems (or situations, or ideas) which average ability students do not recognize (Krutetskii, 1976); and 'learning disabilities' students frequently fail to recognize connections among problems (or situations, or ideas) which appear obvious to average ability students (Lesh, 1980). Qualitative differences in problem solving capabilities are analogous to those that distinguish Piaget's preoperational levels of thought from formal operational levels.

CONCLUSIONS RELATED TO EMPHASIZING AVERAGE ABILITY STUDENTS

Identifying heuristics, skills, and strategies that are used by either 'experts' or 'gifted' students, and attempting to teach them (in isolation, and in an unmodified form) to average ability novices may yield negligable, or even negative, results. It seems plausible that childrens' conceptions of problem solving processes, strategies, and heuristics must develop in a manner similar to the way other mathematical ideas (e.g., counting, whole number arithmetics, rational number concepts) are known to develop. Yet, research has seldom viewed heuristics developmentally. Conceptual analyses of problem solving performance are based on the notion that, to help students' heuristic/strategic conceptions evolve toward mature understandings, educators must be knowledgeable about intermediate stages of the development mathematical ideas and problem solving mechanisms – stages in which problem solving understandings, skills, heuristics are linked to unstable conceptual systems.

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NOTES

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[1] To avoid confusion for readers who do not share my prejudices about the nature of mathematics, and about what it means to *do* mathematics, perhaps I should state another

of my biases. That is, in my opinion, although most mathematical ideas have computational procedures associated with them "doing the procedure" frequently has little to do with "doing mathematics", nor is it necessarily a good indicator of depth of understanding about the underlying ideas. For example, in calculus or statistics, the procedures needed to compute a given derivative or integral or statistic typically bears little resemblance to the network of relationships that underly (i.e., psychologically and mathematically define) the underlying ideas. – Similar statements are true for elementary school mathematics, although they are less obvious.

[2] *All* models have *some* properties that the modeled world does not have, and/or they filter out some properties that the modeled world *does* have. If this were not true, the model would not *represent* the modeled world, it would be *indistinguishable* from the modeled world (Bender, 1978; Burghes, Huntley, & McDonald, 1982).

In conceptual evolution, a 'good' model is one that fits the modeled reality; 'better' models tend to account for more information with more complex relation/organizational systems, with less distortion. 'Stable' models are internally consistent and 'well coordinated' in the sense that (for example) the 'forest' and the 'trees' do not lead to conflicting views.

In this paper, for simplicity, I will describe the evolution of conceptual models using language which suggests that 'stable' (i.e., well adapted) models is a term synonymous with 'good' (e.g., sophisticated, complex) models. In general, in most of our AMPS problem solving sessions, this simplifying assumption was justified. However, conceptual evolution can lead to models that oversimplify or distort reality in much the same way that biological evolution does not always lead to 'higher order' organisms (Gould, 1978). Adaptation may not lead to increasing complexity and differentiation. A cockroach, for example, is a remarkably well adapted organism. – Sadly, many students's mathematical concepts resemble conceptual cockroaches because of the educational environment to which they learn to adapt.

[3] The above description refers to unstable *relational* sub-systems within students conceptual models. Similar phenomena also occur for *operational* or *procedural* sub-systems. For example, unstable procedural systems are characterized by: (a) losing cognizance of overall goals (or the subsuming network of procedures) when attention is focused on individual subprocedures, or (b) mis-executing 'easy' subprocedures when attention is focused on overall goals (or the subsuming network of procedures).

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ADRI TREFFERS AND FRED GOFFREE

RATIONAL ANALYSIS OF REALISTIC MATHEMATICS EDUCATION - THE WISKOBAS PROGRAM

What are the characteristics of the Dutch Wiskobas program and the textbook series grafted onto it?

What is the architecture of partial courses such as column arithmetic, fractions, ratio, measuring, what instructional theoretic frame do they fit into?

Before answering these questions let us give some information on the Wiskobas project!

Wiskobas started about 1970. In 1975 the Wiskobasteam of IOWO (Institute for the Development of Mathematics Education) published an experimental curriculum for mathematics instruction in the primary school (six grades, age bracket 6 – 12). Upto 1980 the curriculum was elaborated in a number of publications. Since the abolition of IOWO the developmental work and the research have been continued in a way albeit at separate institutes. Now, in 1985, the influence of Wiskobas on the commercially available textbooks appears to have been considerable: four out of five schools in our country that have in mind to change text-books, choose now one of the (five) series that have to a considerable degree been influenced by the Wiskobas-work.

The most conspicuous characteristics with regard to contents of the new programs are:

(1) Much attention is paid to basic abilities, elementary mental arithmetic, and estimating.

(2) The basic algorithms, among which long division, are taught and learned as kind of clever calculating.

(3) The subject 'ratio' appears on many spots in the program as an important binding agent between mathematical subjects in the reality; in particular, if relations between quantities and magnitudes are being compared, the proportionality table is an important tool as it is in percentages.

(4) The instruction of fractions and decimals is less formal than it used to be, and the final objectives have been revised.

(5) There is more attention paid to measurement and geometry than it used to be.

The didactical conception, however, is a more essential feature of the

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new programs. This, then, will be the target of our analysis, and for this reason we shall consider the new realistic instruction against the background of other important ways of thought which can internationally be distinguished in textbook series, viz. the mechanistic, the empiricist, and the structuralist views. These four views, or rather, these four teaching theoretic frames into which the textbooks and teacher manueis can be fitted in an ideal typical sense, also reflect globally the basic instructional ideas of teachers (see for instance Thompson, 1984). But this phenomenon will be disregarded here: we are concerned with the global approach of realistic programs and textbook series.

1. A FIRST ORIENTATION ON WAYS OF THOUGHT IN MATHEMATICS INSTRUCTION

Consider the foilowing direction in a fun catalogue of remarkable objects:

This canvas covered with a thin reflecting layer is a marvelous aid for making selfportraits (see Figure 1).

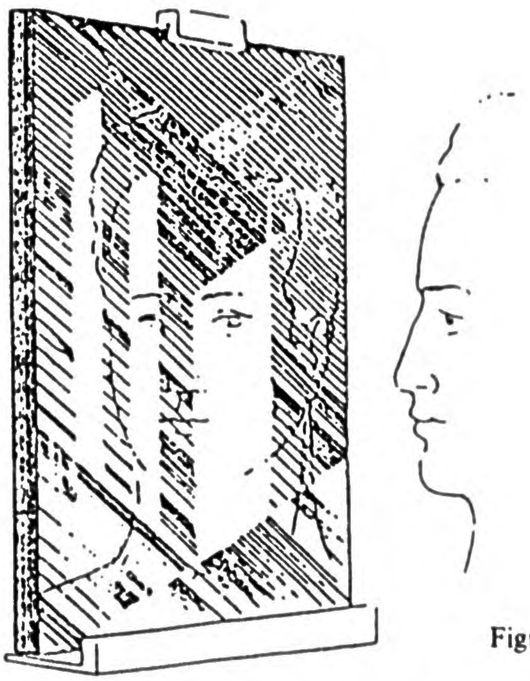

Figure 1.

What went fundamentally wrong when this reflecting canvas situation was drawn?

This question can in principle be put into a geometrical context. The way this would be done according to the various ways of thought in mathematics instruction is an excellent means to charactertise them, as appears from the following.

In the mechanistic approach to the reflecting canvas this query, rather than a starting problem, would be at most an application of previous theory. To start with the rule would be taught, connecting the mirror image and the image perceived upon the mirror. Then the pupils would be allowed to verify the rule and to apply it to appropriate problems such as the reflecting cloth problem.

The structuralist approach yearns for more insight. After having been offered the reflecting cloth problem is schematised and presented to the pupils anew in its geometrical context. This involves that the pupils get little, if any, opportunity to organise themselves the situation mathematically. Much attention is paid to the geometrical elaboration, that is the similarity relation between the images. By means of a new example (how large should a dressing mirror be in which you can see yourself totally?) the pupils can show they have understood the proof.

Both the empiricist and the realistic approach pay much attention to the premilimaries of schematising the reflecting canvas situation. To start with the children are asked to formulate hypotheses on the relation between the images without actually measuring. Natural naive presumptions on this relation will emerge such as that they are equally big, but also that there does not exist any fixed ratio because far-away things become smaller (Schoemaker, 1984). The hypotheses will be discussed and tested or experimentally refuted. The experiment, however, is the source of a new hypothesis: the mirror image and the image on the mirror are as 2 : 1. At this point empiricists and realists have arrived where the structuralists started. Or have they? The schematisation has not been offered but has been developed by the pupils themselves. 'Draw the head of your mate on this pane of glass by looking through'. 'How do you explain the small size of the projection image?' By suchlike draw and aim tasks the attention is fixed on the eye as the centre of the profiling vision rays and eventually they acquire the schematisation of the reflecting canvas problem (Goddijn 1983, Schoemaker 1984). As a matter of fact empiricists and realists differ with respect to the attention paid by the latter ones to the mathematical finishing touch: similarity as the key to understanding the discovered regularity.

The difference between the approaches can also be illustrated by means of the notions of horizontal and vertical mathematising. Horizontal mathematising means in the present case that the problem is schematised in order to be manipulated by mathematical tools. Vertical mathematisation means the mathematical processing and refurbishing of the real world problem transformed into mathematics, that is in the present case proving the relation between image and selfportrait.

In the realistic approach care is bestowed on both components, in the mechanistic approach on neither, whereas the structuralist stresses the vertical component, and the empiricist the horizontal one.

Let us consider this as a first sketch of the ways of thought we have in view. The sketch is illuminating but can also be misleading although not primarily because the typical features of kinds of mathematical instruction have been exaggerated and have produced together a caricature. In fact such procedure is illuminating as long as we are aware of the exaggeration. The proper reason why this example is misleading is the reduction of the fundamental differences to the level of local problem solving. Of course the various global views have their local consequences, which in turn can serve to illustrate the global views by the way of local problem solving examples. Our target, however, is the more global view, problem fields, courses, fundamental long term learning processes, complete curricula. We are aiming at the macroscopic problem solving model, at the global, rather than the local, structure of the solving process. Thus the reflecting cloth problem is not meant as an isolated problem but viewed within a problem field extending from the looking box at the lower grades, via ratio and aiming problems at the higher grades of the primary school to the mirror problems of secondary instruction. Or even more: it is a model for mathematising in its totality.

2. GRADUALLY PROGRESSIVE MATHEMATISATION

We will now illustrate the foregoing sketch of mathematisation, and in particular the horizontal and vertical mathematisation by means of an example of course character. We choose the activity of fair sharing as the red thread through a sequence of well-known subjects: dividing in the lower grades, long division in the intermediate grades, fractions in the higher grades of primary instruction. We will try to uncover the fundamental processes of mathematising in these difficult, controversial, and seemingly worn-out subjects.

The chosen examples have been taken from the Wiskobas work. This means a realistic viewpoint, which is not a monopoly of Wiskobas: fair sharing might be used in this way in any realistic mathematics instruction.

2.1 Fair Sharing and the Division Tables

As everybody knows, fair sharing is a real and motivating activity in childrens' life. What use is made of it in initial mathematics instruction?

What opportunities are created by fair sharing? These are characteristic questions for both the empiricist and realistic thought: looking for starting points in the spheres of interest of children rather than delaying such questions up to the moment when after adding, substracting, and multiplying the art of division gets its turn.

Fair sharing among two participants is an expression of the relations 'more', 'less', and 'equal'. Shares can be compared by one-one-connections, by counting, and by grouping. In addition problems sharing can serve the structuring of numbers below 20 : 5 + 7 = 6 + 6 = 12, it is a well-known fact that doubling is early memorised. Even and odd are related subjects, 'one half' emerges, objects of varying size, value, shape can be cut into equal parts. In brief, as early as the first phase of intitial instruction can fair sharing anticipate on important concepts and structures, though not with the view on operations within arithmetic and formalising division as an operation. This approach is characteristic both of the empiricist and realistic way of didactic thought as compared with the mechanistic and structuralist ones.

Memorising the division tables between the lower and the intermediate grades also happens according to the various ways of thought in a characteristic manner.

In the realistic (and empiricist) approach the division is comfortably embedded in simple context problems where again fair sharing plays its part. As a consequence the bare field of division tables is explored, structured and gradually memorised while using various kinds of arithmetical strategies, where special properties are exploited with the support of formerly memorised facts. When division with remainder is at issue, attention is paid from the beginning onwards to the context dependency of the answers.

What is the use of connecting context problems to learning division (and multiplication) tables?

The children often don't recognise the division operation in the context problem. In order to solve them they use additive or mixed up additive and multiplicative methods. If context problems are chosen as source for acquiring the division concept, the division is, formally viewed, tied to continued adding, continued subtracting, and multiplying. In other words, this integration and association of operations prepares the constitution of the division concept from the other basic operations through progressive schematising and shortening. Division develops from the 'more primitive' operations via informal strategies applied in context dependent manipulations, the corresponding bare' arithmetic of

aiming' multiplications, and aiming estimations. On the other hand, when structuring the field of the division tables, the pupils get the opportunity to take their bearings on the real phenomena of fair sharing, which is particularly important if special properties are to be applied.

This, then, is the way how horizontal mathematisation, that is, gradually learning to identify the division in appropriate context situations, and vertical mathematisation in the sense of ever more skilful and shorter calculating, progress together.

As a matter of fact, the separation of mathematising in these two components is somewhat artificial because of their strong interdependence. But as a means of description the separation is valuable because of its use in characterising textbooks according to the various types – this was clear in the reflecting canvas problem and it will anew show up in the following analyses.

In initial arithmetic instruction the empiricist approach globally equals the realistic one. Only with regard to the vertical mathematisation does a divergence develop, which later on will broaden more and more. Considerably less attention is paid in the empiricist, approach to structuring and memorising the field of division tables, and considerably less time is devoted to keeping up the basic abilities.

The structuralist approach distinguishes itself from the realistic one in the horizontal component of mathematising, which is restricted to the complex of a posteriori applications of the subject matter learned within the formal system. From the structuralist view-point context problems rather than being the source of the division concept are exclusively, or at least mainly a domain of application. As a consequence context problems cannot function as models for operating in the formal system, nor are the formal operations anchored in the informal methods used by children when solving context problems – a disadvantage to application, at least according to our rational analysis.

The mechanist approach differs from the realistic one on all points. Context problems play a minor part, and memorising is not based on structuring and shortening. As a consequence learning and applying the division tables causes serious problems.

It will be discussed later on how this rational analysis is supported by empiricial evidence. Meanwhile it may be posited that the differences touched in the mirror cloth problem manifest themselves equally in the intitial phase of the division course.

2.2. Long Division

The realistic course of long division displays a progression of clever calculating in more or less elementary fair sharing situations.

Let the start be

> '342 stickers are fairly distributed among 5 children; how many does each of them get?'

This at first sight hardly perspicuous situation asks for an organisation of the distribution. In the first phase the stickers are piece-wise being distributed, but soon bigger shares are being dispensed. The process is recorded in a distribution pattern, which keeps up with the distributive activity and supports the next mental steps. Interaction improves and the progress in mathematisation becomes visible (Figure 2).

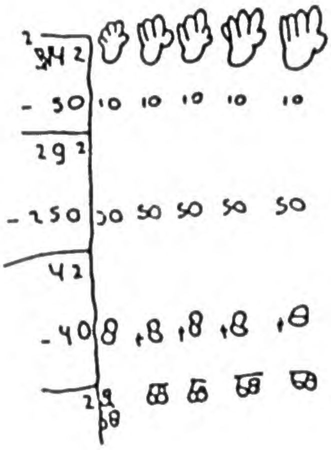

Figure 2.

In the second phase the children are soon satisfied with noting down one column only – 'all get the same, indeed', Other contexts are being introduced, among which that of grouping. After about 15 lessons the children work on different levels (Figure 3).

In the third phase the connection is made to decimals and fractions. Estimating according to powers of 1/10 becomes central but the procedure does not change essentially. Context dependent answers on divisions with remainder are not neglected. Again and again the opportunity is given to invent problems, among which illustrations of bare numerical divisions.

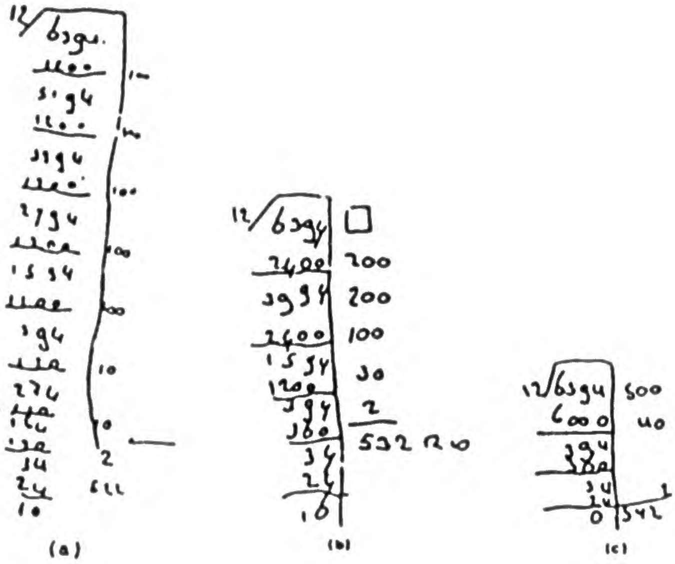

Figure 3.

An example: Given 6394 : 12, invent stories belonging to this sum such that the result is, respectively

532
533
532 rem. 10
532 5/6
532,84 rem. 4
532.833333
about 530

At crucial points in the course it is asked to invent problems and to solve them by a slow longwinded manner as well as by a quick and short one – the pupils should learn to reflect on their learning process and to anticipate on even shorter procedures.

The characteristics of such a course in column arithmetic are: (1) integration of clever calculating in context problems, (2) the progressive mathematisation of the calculating methods, that is, in the present case, schematising and shortening the procedures.

How are courses in long division didactically organised according to the other views?

The empiricist approach equals the realistic one with respect to horizontal mathematisation. The vertical aspect is neglected, that is, the learning process is less directed to the standard procedure as in teaching multiplication less attention is paid to memorising the tables.

The mechanistic view differs in almost every respect from the realistic

one. Column arithmetic is isolated from clever calculating and estimating. Simple context problems are not employed as a concrete orientation basis for the learning of the procedural acts. In stead of increasing schematisation and shortening we notice stepwise complication of the problems – ever larger numbers, more zeroes in the quotient, and so on. In each particular case the definitive standard method is aspired to. A more complex case is not tackled until the less complex one is entirely mastered. Applications are of the *a posteriori* kind. Automatising the procedural acts is predominant. Insight into algorithmising is of secondary importance and only appreciated as far as it efficiently contributes to the process of automatising.

The structuralist approach, is an insightful variant of this isolated course for long division according to progressive complication. From the start onwards they use position material in order to lay insightful fundaments for the procedural acts. As a consequence much less partial cases shall be distinguished and dealt with as separate problems. In other words: the teaching-learning process can be better structured – at least in principle, though the connection between the materialised procedural acts and the mental ones is a bit problematic. As another weakness the formal procedure is virtually separated from the informal manners preferred by children in solving context problems. This means that horizontal and vertical aspects of mathematisation do not support each other – a drawback of artihmetic is to be applied. These are rational objections against the structuralist approach – we delay once more the empirical analysis. Again in the case of long division the differences between the various ways of thought are manifest.

2.3. Fair Sharing and Fractions

Finally we present three Wiskobas activities around fair sharing at the upper level of our primary school dealing with the subject 'fractions'. The first concerns fair table arrangements:

> 'There are 24 pancakes for 36 children. Make fair arrangements such as 2 tables with 12 pancakes each for 16 children...'

The pupils are given a sequence of suchlike tasks. The organising and structuring activities together with the mapping of the situation are again leading the perspicuous schemes, which facilitate communicating and support thinking.

Such schemes allow for extensions and shortcuts. Questions can be

asked such as 'is it a fair distribution?', 'who is getting more': of , 'how much more?' and , 'how many times?' 'What does a tables arrangement look like where everybody gets two thirds of what he gets at ?' 'how can you see by tables arrangements that $\frac{1}{2} + \frac{1}{3} \neq \frac{2}{5}$?'

This opens the road to equivalent fractions, order of fractions; differences of fractions, and it also prepares multiplying fractions. Proportionality tables, with which the pupils are familiar also get a part in tables arrangements.

The second activity of fair sharing in closely connected to the first, although now the division is in fact performed with rectangles, circles, and strips. For instance:

> 'Divide 3 bars among 4 children, draw and describe the parts by means of fractions'.

(The children have already be in touch with fractions – it is stepping somewhere into the course.) We obtain a large variety of denoted and drawn solutions. Some examples $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$; $3 \times \frac{1}{4}$; $3 \times \frac{1}{2}$ of $\frac{1}{2}$; $\frac{1}{2} + \frac{1}{4}$; $1 - \frac{1}{4}$. The problem to distribute 6 bars among 8 children can, if the tables arrangements are remembered, be solved in the same way, but besides these other descriptions are possible: $6 \times \frac{1}{8}$; $\frac{1}{2} + \frac{2}{8}$; and so on. The results of these partitions can be put upon the number line. Circles, rectangles, strips as well as drawings thereof serve as material to be divided.

The third activity continues the second: the pupils are invited to perform productions on the symbolic level if the partition result is given. For instance: $\frac{3}{4} = \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$ or $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ or $\frac{3}{4} = 1 - \frac{1}{4}$. More complex partitions can be generated as follows $\frac{5}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ a standard procedure when each particular object is divided among 6 persons. Then it might follow $\frac{5}{6} = \frac{3}{6} + \frac{2}{6}$ and finally $\frac{5}{6} = \frac{1}{2} + \frac{1}{3}$. If $\frac{3}{6}$ is to be transformed into its alias $\frac{1}{2}$, the tables arrangement can serve as a model: matches , which guarantees the equality of the shares, $\frac{3}{6} = \frac{1}{2}$. The same insight is obtained by a rectangular partition. Every fraction gets, as it were, its monograph, composed with increasing subtlety, while in principle all basic operations are used. At a certain moment the starting fraction is omitted or covered with a blot: ... $= \frac{1}{2} + \frac{1}{3}$. The pupils find aliases and look for the original. Then, for instance, might emerge $\frac{1}{2} \times \frac{1}{6}$, then $\frac{1}{2} \times \frac{1}{3}$ and so on.

These were three fragments of a course on fractions, which arise from fair sharing. We have restricted ourselves: the operator aspect could be

added, for instance by assigning prices or weights to the pancakes or the geometrical patterns (rectangles, strips, and so on). We preferred to give an impression of a few typically realistic activities connected with fair sharing and fractions, rather than sketching a complete course on fractions.

In the empiricist approach less attention is paid to activities on the formal symbolic level than we did in the fraction monographs of the third example, nor do the tables arrangement patterns fit into the empirist approach.

The structuralist procedure stresses the importance of models like strips, rectangles, 'machines', but does not use context problems as situation models of the kind of the tables arrangement problems, nor are open measuring and distributing tasks, such as described in our second example, in the structuralist vein.

The mechanist method for fractions is formalistic and dominated by prescribing rules; none of the three activities above, would fit it.

Thus summarizing; fair sharing connected with fractions again uncovers the differences between the four ways of didactic thought.

2.4. Empirical Analysis

We must refrain from showing and explaining the manifestations of these ways of didactic thought in the textbooks all over the world. For the Netherlands this has been done but the scope of these data is too restricted to be meaningful. Neither is it feasible to illustrate or validate the preceding rational analyses with respect to division tables, long division, and fractions by means of research results in the various domains.

We have, however, indicated in the bibliography a few tens of research studies, from which the following global picture arises, related to the preceding subjects:

(1) Structured learning of tables is more efficient than blind and isolated memorising (see e.g. Brownell, 1935; Baroody, 1985).

(2) Learning long division according to progressive schematisation connected with clever calculating and context problems is more efficient and improves applicability of the procedural actions better than the approach by progressive complication (see e.g. Teule-Sensacq and Vinrich, 1982; Rengerink, 1983).

(3) The tables arrangement, distribution and production activities are useful didactical tools for establishing the equivalence of fractions, developing a fractions language and initiating into

the operations on fractions at the symbolic level (Streefland, to appear).

As a cautious conclusion we may assert that the – still relatively rare – data on the results of realistic instruction in the domains, dealt with above, seem to support the rational analysis. But at the same time let us add the warning that many more data are needed in order to honour the didactical promises of the realistic approach. There are more things at issue than the didactical 'purity' – the teacher and her instruction theoretic frame must fit the method. We have to eliminate these factors for a while from the investigation of the didactic structural characteristics of textbook series and curricula.

2.5. Generalising and Summarising

To be sure it has not been our aim to illustrate progressive mathematisation by spectacular specimens of curricular materials. We meant to explain the realistic conception by means of such examples as we eventually adduced. However, progressive mathematising is also characteristic for curricula in the domains of ratio, measuring, and geometry. Let us make a few remarks on these subjects.

Ratio primarily serves to compare geometrical situations or situations with measurement or arithmetical aspects. It starts visually, qualitatively, informally to become more and more numerical. All kind of models, among which the proportionality table, promote both the perspicuity of the relation and its numerical processing. Gradually the calculations with regard to connections between all kinds of magnitudes are schematised, shortened and subjected to numerical precision.

On the other hand, measuring, for instance of area, shows suchlike phases: fair sharing with qualitative means, comparing by estimates, transforming figures, and finally measuring proper. Measuring by natural units, standard units, refined partial units, by the use of formulas and mappings.

Geometry instruction is similar. No concepts translated from higher level geometry – point, line, plane, planar figures, angles, angle measurement, symmetry, translations, rotations, reflections, vectors – but natural phenomena observed in the spatial environment are at the centre of the instruction. Initially geometric ideas are developed by looking and experimenting – the looking box, photographs, which also serve localising, light and shadow, block buildings. Reasoning and reckoning emerges – views, perspective, the cube, networks of spatial figures, and so on. Fundamental geometric entities such as point, line, and plane are

not defined *a priori*; they arise from activities with light, shadow-light source, aiming, shadow shapes, and projections. In brief, the gradually progressive mathematisation is a general characteristic of curricula according to the realistic conception. This holds not only for primary but also for secondary instruction.

The HEWET project of OW & OC for the upper grades of secondary education is an outspoken example of gradual mathematisation, realised by such subjects exponential growth, differential calculus, matrices (De Lange and Kindt, 1984).

Summarising the characteristics of mathematising on various domains we may say in general: Mathematising is an organising and structuring activity by which acquired knowledge and abilities are called upon to find out still unknown regularities, connections, and structures. The difficulties of mathematising can be of various origin, dependent on the level of the activity. Sometimes the trouble, when organising the problem situation, is finding out the corresponding mathematical aspect (for instance if mirror images are the issue or fair division shall be realised). Another time the mathematical operation itself is the stumbling stone.

Mathematising is a dynamical process: new problem fields must be explored (by anticipating). Then it can happen that the lower ones are used as models of algorithmic character to reach the higher level. The activity of transforming a problem field into a mathematical problem question is called horizontal mathematisation – the problem field is approached with mathematical methods. The activities of processing within the mathematical system are vertical mathematisations. In the horizontal component the road to mathematics is paved by means of model formation, schematising, and shortcutting. The vertical component acts by mathematical processing, raising the structure level in the corresponding problem field. No doubt the separating clusters of activities in the two components looks somewhat actificial. In particular in the realistic way of thought the distinction is difficult to be manipulated, because the transformation and the mathematical processing and structuring strongly depend on each other. Nevertheless the distinction has some descriptive value because of the possibility it creates to typify the global structure of textbooks according to the various ways of thought.

In the realistic curricula much attention is paid, as we explained, both to the horizontal and to the vertical component of mathematising. In other words: the phenomena underlying mathematical concepts and structures in the reality are used both as source and as application. This creates the possibility to orient oneself in the acquisition of mathematical

insight and structure within the mathematical system on the real phenomena. In the structuralist view the vertical component is dominating: operating within the mathematical system is the main part of the mathematical activity. The horizontal component is represented by the totality of *a posteriori* applications of the subject matter learned within the formal system. For this reason real phenomena cannot function as models, to support thinking and operating within the mathematical system. Instead one is working with embodiments of materialised mathematical concepts and structures or with structure stories as concrete orientation bases for the formal operations. In the empiricist curricula, however, the horizontal component determines the global structure. The mechanistic approach is characterised by the weakness both of the horizontal and the vertical component of mathematising. In actual instruction and textbooks 'pure' ways of didactical thought may occasionally be realised but the most usual is mixtures of the four types. Rather than the presence or absence of some component of mathematising we observe the variable stress laid on it. Qualitative descriptions of curricula grafted upon the rational analysis are able to uncover shades and gradations.

3. FIVE CHARACTERISTICS OF REALISTIC CURRICULA

Realistic curricula are distinguished from non-realistic ones on the following points:

(1) the dominating place occupied by context problems, serving both as source and as field of application of mathematical concepts;
(2) the broad attention paid to (the development of) situation models, schemas and symbolising;
(3) the large contribution children make to the course by their own productions and constructions, which lead them from the informal to the formal methods;
(4) the interactive character of the learning process;
(5) the firm intertwining of (related) learning strands

3.1. Context Problems

Context problems have specific forms, contents, and functions. They can be edited in pure arithmetic language, as word and text problems but they can as well be presented by games, plays (dramatisation), stories,

newspaper-cuttings, models, graphs, or by a combination of such information bearers, clustered in themes or projects. By which criteria should a traditional wordproblem meet to become a context problem?

> 'Our car drives 1 to 10 – so we say in Dutch – which means 1 litre of gasoline for 10 kilometers. How much is used for a travel of 234 km?'

It depends on the way this problem is used in instruction whether it is to be considered as a context problem or not. The criterion is whether the intended context is involved in the reasoning and the calculation. Problems in realistic (and empiricist) instruction are intended to be natural and motivating rather than to present varnished preconceived bare arithmetic. Anyway the context must be meaningful: the children are not compelled to forget about their personal experience and knowledge (for instance, 1 to 10 is only an approximate average and in no way precise, driving on the highway is less expensive than in the city, and so on – on the contrary the experiental basis is intentionally used.

Context problems in realistic instruction fulfill a number of functions, to wit that of:

– concept forming: in the first phase of the course they allow the pupils a natural and motivating access to mathematics,

– model forming: they supply a firm hold for learning the formal operations, procedures, notations, rules, and they do so together with other palpable and visual models, which have an important function as supports for thinking,

– applicability: they uncover reality as source and domain of application,

– exercise of specific arithmetical abilities in applied situations.

To stay with fair sharing: a simple distribution task can be the start problem to learn a division tabel or long division. Conversely it can be used as a concrete orientation point if a bare arithmetic problem is to be figured out or a specific property of division as such is to be understood. In both cases the context provides significance, and then to such a degree that eventually the procedure and operation themselves become meaningful and rich context within the formal system. Besides this such a distribution task can function as the application of a previously learned table or of the long division procedure, and as a specific exercise, for instance, in the applied situation of money arithmetic.

In brief, context problems have both a horizontal and a vertical function. On the one hand they serve to make mathematical knowledge and

abilities applicable, on the other hand they lend meaning to formal operating and prevent it from becoming formalistic. In other words; they create the possibility to fill the formal system meaningfully and with riches of context. This explains why we asserted that even bare arithmetical problems can be understood as context problems, namely if their symbol and signal character is strong enough a reference to situations that are meaningful to the pupils. This, however, can more often be realised at the end rather than at the start of the learning course.

Mechanistic and structuralist instruction on the contrary, starts with working within the formal system or concrete materialisation thereof (that is, after a short concrete introduction period); word problems are often preconceived, pedantic, stereotyped. This does not mean disqualifying traditional bare sums, word and text problems as such. Even in realistic and empiricist teaching they play a part as mentioned above. If, however, mathematics instruction almost exclusively consists of this type of problems, it misses, to our view, the aim of applicable concepts, it lacks motivation, and it suffers of a terrifying poverty of meaning. As a consequence for many pupils formal operations within the mathematical system remain as poor of context as the informal operations outside.

3.2. Models and So On

We noticed that simple context problems can function as models – situation models as we called them. In general it may be asserted that character and function of models according to the various ways of didactical thought depend on the relative stress laid on horizontal and vertical mathematising. This explains the use of, among others, situation models in realistic and empiricist instruction, the virtual absence of models in the mechanistic approach and their relatively one-sided use in the structuralist methods, where they function only vertically, that is, as artificially constructed materialisations of mathematical concepts and structures. They are not primarily bridges between the specific occurrences of mathematics in the physical, social, and imagined reality on the one side and the formal system on the other – take as an example, in the case of division, the MAB material, in the case of fractions, the so-called machines, which are mainly functioning within the mathematical system. Our three examples in teaching fractions show, on the contrary, the sublety of the relation between context problem (situation model), schema, and symbol. The usual mistakes with respect to the order of fractions, which are not at all eliminated by materialisations, are

virtually excluded by the approach sketched above, as appears from Streefland's investigations. The power of a symbolism fitting a situation model is obvious in the example of the tables arrangement. A similar remark can be made with respect to learning long division. In both cases the situation models function both horizontally and vertically. Actually the same holds for many other models, in particular, these related to measuring such as the number line and the strips, which are also used in other approaches than the realistic one.

3.3. The Pupils' Own Production and Construction

In our examples adduced above the pupils' contribution to the teaching course was considerable. In the case of long division the pupils' informal methods in solving context problems of distributing and dividing were the direction posts on the road of gradual construction of the algorithm. Likewise notation patterns and shortcuts are pupils' invention, for instance estimating the number of tens and hundreds in what becomes the result of the division. Moreover there are a great many open construction tasks, such as composing stories for a division problem that is proposed with several results; the analogue in the case of fractions is producing various partitions if the result is given, variegated tables arrangements, fraction monographs and so on. To and fro pupils produce *simple, average,* and *difficult* problems for the teacher to be used as test problems, This, then, means reflecting on their own learning process, and as far as the difficult ones are concerned; sometimes anticipating on concepts and procedures to be acquired in a near future. Constructing, reflecting, anticipating and integrating are fundamental functions of the pupils' own production. Seen through the teacher's eyes they are diagnostically valuable.

In structuralist didactics production is no essential element. On the contrary after a preparatory, maybe playful, phase there is little room left for informal methods and their gradual transformation into formal ones: it is instruction in a straight jacket. This is even more true of mechanistic instruction.

3.4. Interactive Instruction

The constructive character of mechanistic mathematics is embodied by textbooks for solitary arithmetical work and paper guided instruction. Realistic and empiricist instruction, however, because of their search oriented approach, ask for active contribution of the pupils as stated above. To say it more forcefully, the pupils informal methods are used

as a lever to attain the formal ones. Such a method requires explicit negotiation, intervention, discussion, co-operation and evaluation (Bacomet, 1985), thus a specific didactical shaping of interactive instruction. In programs and textbook series espousing the realistic thought the differentiation pattern is adjusted to the necessity of co-operation. The teacher manuals contain practical suggestions to shape interactive instruction.

3.5. Intertwining of Learning Strands

Only briefly shall we tackle the intertwining of learning strands with a view on the preceding. In the initial instruction fair sharing was connecting to counting, adding, memorising and much more. Learning long division took place through context problems, clever calculating and estimating. Parts of instruction of fractions were intertwined with ratio and measuring. We indicated ratio as an important binding agent between numerous subjects and domains and the reality. Measuring offers a natural access to calculating and models (among which the number line) covering a broad field. Geometry yields problems that connect almost all subjects of arithmetic and measuring.

This, then, was the fifth characteristic of realistic curricula.

4. THE INSTRUCTION THEORETIC FRAME

With this sketch of the macro structure of curricula organising the long term learning process we also intend to discern sharply the theoretical frames in which the various programs are embedded.

Now we shall pay some attention to the constructs that are the most important, from the viewpoint of teaching theory, the structuralist, and the realistic ways of didactical thought.

First we consider the two views under which mathematics as such is seen by both of them. Then we pass to corresponding instruction theoretic frames. After a look to the connection with general cognitive theories we shall conclude with recommendations.

4.1. What Is Mathematics (Instruction)?

Under a structuralist teaching view mathematics as a school subject is a compendium of acquired structures, concepts, and procedures of thinking. It is a preconceived structure, a completed building, something that is given and needs no exploration on the teaching level.

Within structuralist instruction one could globally distinguish three

variants. In the first symbols are assigned a 'meaning'. After a concrete introduction they soon acquire a meaning for the pupils: the formal system is a syntactic system. Gagné in his more recent work (1983) could be considered as a representative of this interpretation, although mechanistic features are not lacking in his position. According to the second variant the formal system is materialised by embodiments of the mathematical structure. Dienes and his more moderate followers are representatives of this variant. The third variant tries to make the formal system accessible by structure stories: Frédérique Papy has lines singing, number pairs dancing, arrows tying relations, and symbols performing mathematical operations.

The main realist objection against the structuralist interpretation of mathematics is its start from acquired insights and concepts rather than considering structures and concepts as goals that are to be acquired. Realists view mathematics mainly as a human activity, which at each age and level may lead to (un-)valuable mathematical performances and (wrong) products. They knock mathematics off its pedestal. They – these are among others mathematicians of big renown: Thom, Hilton, Whitney, Lakatos, and Freudenthal. Thom attacks the formalism, the premature introduction of mathematical concepts; he is pleading for geometry. Hilton stresses the constructive and productive element and long term learning processes. At the same time he draws the attention of the researchers in the field of mathematics instruction to the necessity lively to understand the dynamical process of learning mathematics, thus to experience the very nucleus of the mathematical activity. Whitney too stresses these elements as well as the selfconstraint the teacher should observe when explaining problems. At the same time he stresses the necessity of meaningful teaching, meaningful and motivating problems, non-pedantic and non-formalist. Lakatos fights the dogmatics view on mathematics as eternal and unassailable truth, which suffocates search, failure and adventure. According to him textbooks should be rewritten in order to reflect the dialectics and the growth of mathematics. Heuristic instruction need not follow precisely the historical course of the intended concepts; it is rather a rational reconstruction of the historical process with the view on the learner, Lakatos claims. Freudenthal in his 'Didactical Phenomenology of Mathematical Structures' says: '... that the young learner is entitled to recapitulate in a fashion the learning process of mankind'. Here again we notice the historical orientation completely lacking in the structuralist thought. Rather than from material and materialised 'embodiments' Freudenthal starts from phenomenal em-

bodiments, the real phenomena behind the mathematical structures in order to have the concepts or, as he says 'mental objects' constituted (Freudenthal, 1983).

4.2. Specific Instruction Theoretic Framework

One might consider Van Hiele's level theory as the first example of a specific instruction theoretic framework of realistic mathematics instruction (Van Hiele – Geldof and Van Hiele, 1985). There the didactical necessity of phenomenological exploration at the ground level is stipulate, where it should precede the formal mathematical operations on the first and second level. Certain instructional phases are required to attain higher levels.

It is a remarkable fact that, formally viewed, this level theory agrees with the first specific instruction theory of the structuralist kind, to wit Dienes' fase theory of the learning cycles. 'Formally viewed', I said, because the material differences are considerable. At Dienes' ground level the pupils get in touch with geometrised mathematical structures, with 'embodiments' of concepts in the form of games in artificially constructured environments, whereas Van Hiele's ground level the pupils investigate the aspects of reality for which the mathematical concepts and structures (can) serve as means of organisation. 'Embodiment' is here 'embeddedness'; the object of organisation is here a natural rather than an artificial stuff; rather than mathematics translated to a lower level in an artificially created environment it is in Van Hiele's case the natural-fysical, social and imagined world that provokes an activity which aims at structuring – that seems a reasonable interpretation of his level theory. We shall not tackle the problems we meet in this theoretic frame if we step outside traditional geometry instruction. Nor shall we discuss the trouble Dienes himself had with his learning cycles, for instance, in instruction of fractions. The only thing we had in mind was the uncover the early roots both of the realistic and structuralist instruction theory trees. Let us, however, add that recent signals indicate a growing interest in specific instruction theoretic frames, after the calm seventies (Steiner, 1985).

From the preceding one might conclude that in order to construct an instruction theoretic frame we must view primarily progressive mathematisation in long term learning processes, and connected with it, the structuring and sequencing of courses in textbooks series, particularly of realistic signature.

Of course one may ask whether in actual teaching the contrast between

the structuralist and the realistic organisation of instruction is as sharp as sketched above. Anyway a comparison of textbooks published in The Netherlands linguistic area (Dutch and Flemish) reveals enormous differences: two fundamentally different views on and realisation of mathematical instruction. In other words: two enormously different instruction theoretic frames (a picture which becomes even more confusing if we add the empiricist and mechanistic view). This is also reflected by instruction psychological research, in particular with respect to its two pilars (Skemp, 1984), that is diagnostic interviews and teaching experiments. However, more often than not does the specific instruction theoretic frame remain implicit; it is neither discussed nor even uncovered. A most interesting example for this case is 'The psychology of mathematics for instruction' by Resnick and Ford (1981), and the comments on it, which vary from 'this is the true shape of mathematics instruction to be envisaged' to 'this is a travesty'. Of course this has much to do with the respective choice of instruction theoretic frames such as we tried to sketch specifically for mathematics.

4.3. General Instruction Theoretic Frames

Our last remarks were meant as a transition to the question of what is the relation between the specific and the more general learning theories, such as the activity psychology and the cognitive information processing theories. Well, at a closer look both the basical conceptions of structuralism and realism can be fitted in both of these theories. Elsewhere we signaled the same phenomenon with respect to Gal'perin's theory of the stepwise formation of mental acts, but the same may be claimed with respect to other general theories. Obviously these theories leave much open space, though of course there might be limits. For instance, Gagné's former cumulative learning theory can hardly be reconciled with the basical ideas of structuralism – and certainly not in its concrete realisation – and in no way with the empiristic and realistic view. This, however, does not hold for the cognitive learning theories. Indeed, they can include both frames specific for mathematical instruction.

Conversely this means that starting with a general learning theoretical framework such as the 'information processing learning theory' or Galperin's activity theory about the stepwise formation of mental actions, or Davydov's conception of learning theoretic concepts, one can arrive at completely different constructions of instruction. In brief, these general theories are no construction theories, because no specific rules of

action or prescriptions can be borrowed from them for the construction of instruction, and in particular of courses. They alone cannot be a basis for developing textbooks and the design of teaching experiments. There is more to it, and this lacking element should also rationally be accounted for. For instance, why in psychological research one relies on the 'sequence of mental activity', or the relation between material, verbal and mental actions, or the 'semantics' of manipulating structured material; and why little attention is paid to conceptions like 'meaning' and 'intention' (Carpenter, 1985) – particularly neglected in the information processing theoretic frames – whereas in general cognitive research the crucial importance of meaningfull situations, experiences, intuitive notions and context has emerged for the retaining of facts, procedures, of logical reasoning and of concept formation (Anderson, 1980).

It is remarkable that research on initital instruction often shows realistic features (Carpenter and Moser, 1982, Ginsburg, 1983).

The realistic instruction theoretic frame is, as it were, cast in the same mould as the first phase of intitial instruction itself, and based on the same ideas. In other words, the realistic approach towards fractions is similar to that towards number in its first stage – broad phenomenological exploration, stepwise mathematisation, consisting of schematising, shortening, structuring and increasing numerical precision. As a matter of fact this also holds for other subjects. It is the very nucleus of the instruction theoretic frame of realistic mathematics instruction, quite simple as a theory but quite complex with regard to its realisation, as experienced by Wiskobas. But for sure, a theory with vast practical perspectives.

4.4. Recommendations

To our view it would be recommandable if in research on mathematics instruction in general:

(1) More attention would be paid to rational analysis where in particular the viewpoints and arguments from the four instruction theoretic frames are being confronted with, and weighed, against each other.

(2) More comparative empirical analyses would be undertaken to match these rational analyses.

(3) The scientific efforts would not be focussed on micro cognitive processes but long term learning processes, as taking place in courses, would be included in the research.

As a consequence one should influence the analysis and development

of textbooks. In our country this has been done systematically on the primary level. There are now textbook series cultivating more or less realistic mathematics instruction on the primary level. It should be added, however, that at the present moment the conditions to realise the most recent programs such as intended are far from optimal. Also in this respect we should be realists and abstain from drawing Wiskobas' selfportrait twice as large as it really is on the reflecting canvas at Noordwijkerhout.

NOTES

[1] The former IOWO included also the projects for secondary instruction, Wiskivon (12 – 16) and Hewet (16 – 18), which will virtually disregarded here, although they too influenced mathematics instruction in the Netherlands considerably.
[2] This problem fits into secondary instruction. It has been borrowed from work of A. Goddijn and G. Schoemaker, collaborators of the former Wiskivon-project, now at OW & OC.

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HASSLER WHITNEY

TAKING RESPONSIBILITY IN SCHOOL MATHEMATICS EDUCATION

For several decades we have been seeing increasing failure in school mathematics education, in spite of intensive efforts in many directions to improve matters. It should be very clear that we are missing something fundamental about the schooling process. But we do not even seem to be sincerely interested in this; we push for 'excellence' without regard for causes of failure or side effects of interventions; we try to cure symptoms in place of finding the underlying disease, and we focus on the passing of tests instead of on meaningful goals.

This is true in particular with the two national commission reports of 1983; both call for raising standards, taking more mathematics in high school, and increasing the school day and year. In *A Nation at Risk* [1] we read "Students in high schools should be assigned far more homework than is now the case". And later, "To Students: You forfeit your chance for life at its fullest when you withhold your best efforts in learning..." I will show below how these students are unable to understand what school math is all about (with very real cause); they are now wrongly judged as "withholding...efforts" and are demanded to do far more of the same work. This cannot but throw great numbers, already with great math anxiety, into severe crisis, with increasing drop-outs, delinquency, crime and suicide resulting; compare Albrecht [2].

There is also great pressure to start teaching children at an earlier age (in Minneapolis 11% of the kindergartners failed their tests and were required to take remedial work in the summer of 1983) [3]. No regard seems to be paid to the possible destructive consequences to the beautiful natural growth of the children [4,5].

In brief, caused in good part by the present difficult times, I see our children having an extremely ineffective experience, and am calling for an armistice, to enable local school districts to look for better ways in more peace; this is the present pressing need. Next, moving to truly better mathematics education requires a full and proper study of all the complexities of the situation, socio-political as well as intellectual; I give the

main parts of such a study here. It will include a natural type of cure that requires a change in attitudes but is perfectly feasible in the classroom, if we can just allow it to take place.

CONTRASTS IN LEARNING

What is school for? There is a strong call for discipline. Is it for the moment, or to extend to the future? In the latter case, it must be meaningful in each student; it must finally come from within, growing into real responsibility. And if this student is to help the U.S.A. technologically, his or her real human powers must grow through the school years. In mathematics, this means learning to find his or her own way through problems of new sorts, and taking responsibility for the results!

In most schools, this has been pretty completely forgotten; the pressure is now to pass standardized tests. This means simply to remember the rules for a certain number of standard exercises, at the moment of the test, and thus 'show achievement'. This is the lowest form of 'learning', of no use in the outside world. For an example, the exercise (stimulus) 6)608 requires the response 101 R2.

I shall show the reality of these statements, partly from results of the second mathematics assessment of the National Assessment of Educational Progress (NAEP) [6]. I wish to compare *school learning* with *learning from life* (as apprentices did in earlier times). First, take an example from life.

Joan and Lewis are helping plan a garden in the country, rectangular in shape, 10 feet long and 6 feet wide. Because of rabbits, they wish to put a fence around it. How many feet of fencing do they need? Of course they can think "10 ft and 6 ft and 10 ft and 6 ft". But the garden being real, they look at where the posts should go, and realize that the fence must be away from the edge of the garden so that they can walk inside it. Then they also laugh to think they had forgotten the need of a gate!

In the NAEP, the question "how many feet of fencing" are needed to go "all the way around the garden" was asked of 9-year olds and 13-year olds. Three answers and the percents of children choosing these answers are shown in the table.

	age 9	age 13
32 feet	9	31
16 feet	59	38
60 feet	14	21

Why did not all the children get the 'correct answer'? If they were interested, involved in it as a *real problem*, they of course could have drawn a picture or *made it real* in some way, and looked to find the answer. But that more popular answer, 16 ft, shows the main reason: In a *school problem*, you just guess what operation to use with the given numbers. For adding, you get 16 feet; for multiplying, 60 feet. In other words, in school, you don't look at meanings; you know the problems are not real. (And, in contrast, Joan and Lewis noticed that 32 ft was not correct.)

That so few of each age group got the 'answer', in the broad national sample, shows strikingly how schooling *does not serve* to help children see reality.

Suppose you are a high school student, making arrangements for a tournament of a bowling league. There are six teams, each to play each other team just once. So, calling the teams $A, B, \ldots, F$, you list the games to be played. Can you let several games be played at the same time? Probably yes. By the way, how many games are there? You count them in a jiffy.

This is real life. And school? "How many games" was asked of 17-year olds in the first two mathematics assessments. Just 4% and 5% respectively of the students got the right answer. This was school, and they had not been *taught* how to do it, hence 'could not' do it! (Very likely many thought: 6 teams, each playing five games, makes 30 games. This is *making a stab* at it, but definitely is *not* taking responsibility for getting the right answer.)

How about those 'skills' taught in school? The NAEP editors say [6], p. 145, "Students demonstrated a high level of mastery of computational skills, especially those involving whole numbers". Let us look at some of the NAEP evidence.

The exercise $\begin{array}{r} 21 \\ +\ \underline{54} \end{array}$ was given to 9- and 13-year olds; 10% and 2% respectively missed it. In the form "what is the sum of 21 and 54?" the percents of failure were 31% and 9%. What should be made of this? Did these children *not know* (even at age 13) that the 'sum' was the result of adding (in school terms) the two numbers? Were they not *taught* that they were the same? Or does teaching not give learning? Whatever the cause (see taboos, below), the 'skill' is only *partly usable*, so certainly by many students *not* with responsibility.

In most schools, you are taught algorithms, never with the suggestion that you try *thinking through* to an answer. And the result? All three age groups (9, 13, 17) were asked to "Subtract 237 from 504". Correct answers were given by 28%, 73%, and 84%, respectively.

Mental arithmetic used to be common in schools. Here, you could think: 'Take away 7, brings you down through 500 to 497; take away 30 leaves 467; take away 200 leaves 267". Or, from 237 through 240, 300, and 500 to 504 gives the answer quickly.

The exercise 6)608 had the success rates 69% and 65% from the 13- and 17-year olds, respectively. Of course, to divide six \$ 100's and eight \$ 1's among six people, with two \$ 1's left over, is wholly trivial; but that is life, not school.

I do not call this mastery; I call it failure. These questions are in the very elements, and all school children should be able to think them through with ease and *solid confidence*. Schooling has definitely broken down, even on those skills that are constantly emphasized.

EARLY CHILDHOOD

We know that very young children explore their environment and learn in manifold ways, at a rate that will never be equalled in later life; and this with no formal teaching. It is through their play that they experiment, see interrelationships, get some control over their surroundings. Learning to walk, manipulate things, sense parents' feelings, communicate verbally and non-verbally, are all quite complex and often subtle. Two-way verbal communication is especially complex; hearing a succession of sounds, decoding it into a message, considering one's wishes in regard to it, putting a return message into verbal form in proper order and expressing it are all done in a flash. Anything taught in a school classroom is extremely simple in comparison. Yet these same children *seem* unable to learn most of those trivial things, especially when *taught*.

Preschool children know a lot about small numbers that we are quite unaware of. Laura has five kittens; she knows well the varying patterns as some or all of them move about, though she does not think consciously about these patterns. Coming into the room and seeing three kittens, her immediate thought may be "find another kitten, then find the other one". She has two quick equivalent thoughts which we might express as seeing 5 as 3 + 2 and as 3 + 1 + 1.

The broadness of their learning is certainly largely due to their complete freedom to think in any direction at any moment; curiosity, imagination and flexibility are keys to their rapid progress. They learn flexibility; things move, changing apparent size and shape, words take on new meanings, things happen one way one time and different ways other times.

I told Jonathan, age 5½, a story (to be quoted later) about twenty-six children going on a trip. There will be cars to take them, each car holding four children. How many cars are needed? I first suggested fingers as children. He started bunching his fingers into groups of four; but there were no cars handy. So I drew him a rectangle for a car, and put a circle (a child) inside. He put in three more, and counted all four. Now he drew another car, put in four children and counted all eight, and continued, putting two children in the last car and counting all 26. But the cars must go (he apparently thought), so he put four wheels on (one side of) each car, then linked them together, forming a train. He had brought the story to life, forgetting my original question of "how many cars?" But when asked, he counted them, finding seven.

Numbers becomes a *tool* when you use them for a purpose. In a class of six-year olds (in Brazil) the teacher was explaining how to find 3 + 5 by drawing ducks on the board, not noticing a boy in the back of the room saying to another "yesterday I gave you ten cards; now you have given me seven, so you still owe me three".

Children *process* information far beyond what we realize. In the early sixties, some mathematicians tried teaching third graders about the transitive law: If A is more than B and B is more than C, then A is more than C. The results were somewhat dubious. The children were ahead of the teacher at the start, as found by Trabasso later. He tried *testing* children, age 4 and higher, on their *use* of the 'law'. A typical form of the experiment was as follows [7]. The child first learns the names of six children from pictures, which show head and shoulders only. The experiment has the pictures (hidden) arranged in order of decreasing 'height', A, B, C, D, E, F. The child is now shown, in random order, just the five *adjacent* pairs $A-B$, $B-C$, ..., $E-F$; for each pair, the child is told which is the taller and which the shorter. When the child has learned these well, he or she is tested on all fifteen pairs, being asked which is taller (or shorter). The time taken for the child to respond to each pair is recorded.

To respond to the pair $B-D$ for instance, one knows: B taller than C, C taller than D; so B is taller than D. Pairs far apart in the sequence take more uses of the transitive law.

Thus the pairs far apart can be expected to take the child longer than those close in the sequence. Did this happen? No, just the opposite. In fact, the original adjacent pairs tended to be the hardest of all! What this shows is that the children *did not* (in general) just learn what they were taught; they *used* the information to get a sense of the whole group in order of height; hence those far apart in the sequence were thought of

as far apart in height, hence easy to see as such. This is fine sharp reasoning, done without any suggestion.

Many examples of how young children think about numbers on their own can be found in Ginsburg [8]. This natural talent and learning is something valuable to be *further nurtured*; then the children can grow with joy and speed. We will see how these powers get more and more suppressed in school.

HOW THE PROBLEMS ARISE

How can it be that when preschool children think so naturally and successfully, in school they get pulled into dropping such thoughts and trying to think only as they are told? We will see that this is basically through an interaction of attitudes between teacher and child, leading inexorably to this result in the present climate of schooling. Because of this, the childrens' natural thinking, with looking for meanings, becomes gradually replaced by attempts at rote learning, with disaster as a result. And the more the pressures are applied to enforce learning, the more its rote character is fixed, resulting in further failure. The attitudes are firmly fixed in high school; and for this reason, attempts to improve high school learning are essentially doomed, if the changing of attitudes is not undertaken in a basic manner.

Entering first grade is pretty sure to bring deep-seated feelings to most children (even if these feelings remain hidden). "Now I am in a real school, and must learn the right way" is a natural thought. Thus the children are at risk at the start.

The early experiences with simple stories with mathematical elements normally go well, and with exercises like $2 + 33 = \square$, the children find easily how to write in a number. But because the exercises below are on standardized tests (since they knock out a large percent of the children easily), the vestiges of the 'new math' like

$$\square = 5 + 3, \quad 4 = \square - 7,$$

remain. The first is commonly thought of by the children as "written backwards", with the answer on the wrong side. The second is totally incomprehensible to them; they can only guess what you do with the two numbers. This forces the teacher to try explanations: "The equal sign means..." which is jibberisj to the children (especially since they had always done their own thinking and can't comprehend 'think this way'). (The mathematicians had intended such exercises merely as exploration

and hence better understanding of the equal sign; but this misses both the world of the young child and of the publishers; the latter must tell the teachers how to teach and test everything.) The failure of explanations is beginning, and with it the attitude, "just try to learn the rules for the day".

The failure of the traditional teaching process begins with addition and subtraction of two-digit numbers. As a typical example, the teacher may begin a problem with "The farmer has 37 baby chicks, and sells 14 of them. How many does he have left?" Then comes "We write the 14 under the 37, like this, and work first with the units. What is 4 from 7?"

The message to the children is: Now you drop the *meanings* of numbers, and just look at the *pattern of the digits*. Moreover, you shift your point of view from one problem to two,

$$\text{from}\quad \begin{array}{r} 37 \\ \underline{-14} \end{array} \quad \text{to} \quad \begin{array}{rr} 3 & 7 \\ \underline{-1} & \underline{\;\;4} \end{array}\,.$$

Both the new problems are done easily. So the teacher feels good, and gives lots of problems of this sort to do, to drill the children well; at least one part of the curriculum is 'mastered'. The teacher is of course following the standard practice, time honored and followed everywhere.

Next comes a rude shock. "Now I will teach you how to do *subtraction with borrowing*". With apples, there were 42 to start, and 18 were sold. Again the children see the problem broken into two,

$$\text{from}\quad \begin{array}{r} 42 \\ \underline{-18} \end{array} \quad \text{to} \quad \begin{array}{rr} 4 & 2 \\ \underline{-1} & \underline{\;\;8} \end{array}\,.$$

But now the teacher continues with, "Can we take 8 from 2?" "No!" in a chorus (you take 2 from 8). But the teacher says, "We must borrow from the 4. What does this 4 mean?" The children are in a quandary; the old way, 2 from 8 is 6 and 1 from 4 is 3, is easy and nice, but now you must do a funning thing that is hard to learn.

To the teacher, there is a different stimulus: now the 8 is bigger than the 2. But to the children, the stimulus is the same: a pattern of 4 digits, just as easy to do by the old method. There is now not only complexity, but conflict besides, and this is a sure start towards math anxiety.

I must explain those words 'rude shock' above. It does not show on the faces of the children. It is a message to most of them that they are in for hard times ahead, with changing rules; school math will not be so easy and pleasant. They foresee a lot of failure; and this becomes a fact, as we know. I will speak more of this later.

Children (or adults) commonly say they liked math to a certain point, when they started losing out; perhaps with long division, with fractions, or with algebra (or calculus ...). That was probably the time when they completely gave up on meanings. The stage of learning mostly by rote usually began much earlier; however, they could still sense that they were getting along and could pass tests (but would not be able to explain meanings). Since my moving into intensive work in classrooms (and out) in 1967, the majority of students I have had contact with (in the thousands) were quite clearly in this intermediate stage. A very critical point is the shift from seeing meanings to learning the rules for the day.

With increasingly complex situations, long division and fractions in particular, the gap grows larger, the pressure from the teacher to *do many problems* increases, and with the added time pressure, rote learning becomes the only way out.

There is at present great pressure to teach problem solving. How is this going? In elementary school, this translates into 'word problems'. Using these in the simplest stories was fine in first grade. But later, when rote teaching and learning is well underway, it no longer works; both teacher and child dislike them, and they are mostly skipped.

The cause is fairly easy to see. Take an example above, where the 6-year old had given ten cards, gotten seven in return, and said, "You still owe me three". The only complexity is in the several elements, giving and receiving, before and now, having and owing. How can the teacher *teach* all this? This will involve considerable explaining of the different elements. By third grade, those children learning by rote will largely reject explanations, only wanting to know "how to do it". The teacher knows this, and knows that the explanations will be only partially successful; being under time pressure, she would much rather leave it to those children who can work it out. Of course *true problem solving* is looking at a new situation, exploring it, organizing preliminary findings and so on; *teaching* this misses the basic fact that in real life, the stimulus to the worker is the situation itself, with no teacher present. The normal high school attitude, "Just tell me which formula to use", can never lead to problem solving.

Can children solve 'problems' on their own? Of course, if allowed and encouraged, as Jonathan was with the "26 children, four per car". But as a *school problem* it is far away from the normal rote learning in later grades. The question, how many cars are needed, was asked of 9-year olds with calculators in the NAEP. One out of an average class of 30 – 35 children (3%) got the correct answer, 7 cars. For more information, 12% chose the answer 6.5, and 7% chose 65.

For one more example, take the problem, "One rabbit eats 2 pounds of food each week. There are 52 weeks in a year. How much food will 5 rabbits eat in a week?" Correct answers were given by 47% and 56% of the 9- and 13-year olds respectively. This is clear *irresponsibility*, not looking to see what was asked.

These examples furnish further evidence that, right through the elementary school years, the attitude that "school math is something for itself, not for life outside school" is becoming entrenched. In high school, algebra is the principal mathematics topic; it is too abstract and unused (in spite of pictures of bridges, etc., in the texts) to change the impression that it is just for school purposes. The resulting attitude, "Just tell me which formula to use (don't ask me to think)" is well-known as the norm. Of course learning algebra with this attitude is slow and difficult, and useless for applications; only for the best students is algebra a sane course to take (unless attitudes are changed). Being told to "take more mathematics in high school; your future depends on it" is a false and harmful message to give the failing students; only with sharply changed attitudes can mathematics regain its proper place.

PRESSURES ON TEACHERS AND STUDENTS

Teachers are commonly blamed for the failure of their students. This is an unwarrented attack on teachers; they are caught in the 'system', in the elementary school in ways which we describe further, and by the continuation of attitudes in high school. The demand for "better teachers and teacher training", by most professionals, will make little dent in the system or in students' progress if the system remains basically unchanged.

We have seen how teaching begins to fail in early grades. Beyond the first grade or two teachers become used to teaching and reteaching the same topic; they come to believe that most children cannot learn except by continuing this indefinitely. (And each new teacher must reteach what the last teacher failed in.) The real difficulty is simply that most children get mixed up in and forget the rules for the particular day; but practically no one realizes to the full this stark fact, or simply hides from it. (It is too threatening to think that teaching for meaning, which teachers *try* to do, is failing completely for most students.)

The obvious result of this great time loss is continual pressure on the teacher to "cover the material". Under this pressure, more drill and more homework (or class work) is given, and the children, under this increased pressure, naturally react by focusing still more directly on the

rules for the moment; this may result in immediate gains, but also in repeated forgetting as the different rules get piled on top of each other in a jumble. The resulting further failures puts still more pressure on the teacher, and the vicious circle is in full swing.

The present demand for accountability, with increase focus on passing tests and monitoring the students constantly, build up the same pressure further; meanings get thoroughly dropped from view (under behaviorism they play no role anyway), school math becomes nothing but a giant mass of meaningless splinters, and there is intellectual chaos. For instance, I have seen countless hours of teaching addition and subtraction of "mixed numbers", in later elementary school and into high school, with momentary success and extended failure. Leka, in a high school remedial program, learned how to do $6-2\frac{1}{3}$, but was stuck on $8\frac{1}{2}-8$. When egged on (by me) to find the answer through money ("Now take away that \$ 8") she found the answer, $\frac{1}{2}$, but did not dare write it without the assent of the teacher. And another NAEP exercise, $2^2/_5 + 5$, was done by 43% of the 13-year olds and by 65% of the 17-year olds; yet it requires merely knowing the meaning of a mixed number. It is a common professional attitude that if the teacher can get the student to write a correct answer, the job of the moment is done (there is 'achievement', as required under behaviorism). Is this *learning for future use*?

The pressure on the teachers naturally pass onto the students. The climate becomes more strict. "Do what I say, and nothing else" is a general message. Taboos arise easily, for instance: "Don't use your fingers". "Don't guess". "Don't try things to see how they work". "Write nothing but the answer". "Don't speak to your neighbor, that's cheating!" Along with "Pay attention constantly to the teacher", the general message is "No thinking on your own!" Thus attempting to *assimilate* what is taught or discussed is ruled out; the students isn't given the chance.

The demand for constant monitoring of the students by the teacher has further effects. She (or he) must reach as many students as possible each day; hence must ask only rather trivial questions, and wait at most a second for an answer before turning to another student; and only the right answer is listened to. This effectively rules out thinking before speaking, further solidifying the system.

Thus the main effect of present-day pressures for student achievement is to stop thoroughly any meaningful learning; "problem solving" is left far out of the picture. Students and teachers are all victims; math anxiety and teacher burnout are inevitable consequences. Thus the effect of the

recommendations from the two national commissions of 1983 about more math and more work will be the *opposite* of what was intended. And it is the superficial character of the reports that allows such happenings. As I see it, this country, on the highest level, is not yet taking proper responsibility for the growth of its youth, its most important asset. This shows up starkly in the passivity and dullness of high school students, so usual at present [9] .

COMING ALIVE

We have seen how preschool children are surpassing us all in their way of exploring and learning as a natural process, including reasoning in logical and mathematical directions; also how the demands for long and tiresome work to show a myriad of isolated bits of knowledge has reduced the school population, right into high school, to skeletons without flesh or blood to render them human. No wonder this country is beginning to "lose out technologically"; high order progress requires live human powers, going in many kinds of directions and *gaining control* over varied domains and their interrelations.

Under Benezet (Superintendent of Schools in Manchester, New Hampshire) in the thirties, whereas children in regular early to middle grade classes were not ready to admit they might have read anything, those in the experimental classes, where open discussion for the improvement of language skills and promotion of reasoning skills was the order of the day, were alive with descriptions of what they had read. And though they were taught no formal arithmetic at all, the experimental classes surpassed the others in both mathematical problem solving and writing skills. Should we mention that these children were mostly from the immigrant district [10]?

One can go beyond Benezet by also promoting natural discoveries about numbers and related matters. A good *number sense* renders the usual school math with its 'computations' a rather simple and small domain of knowledge in contrast.

As a natural extension of preschool learning, one can explore in many ways the beauties and complexities of our decimal number system. At first, sets of things are explored (not taught!). Looking in the egg box, four eggs are gone. Look; there are two fours left! You *see* the thirds of

twelve, so in learning the word 'third', the *meaning* comes first, the *language* after, which is the natural and proper order. You need never forget 'thirds' again. Moreover, you see sixths also (each one is half of a third; language: $\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ or $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, etc.).

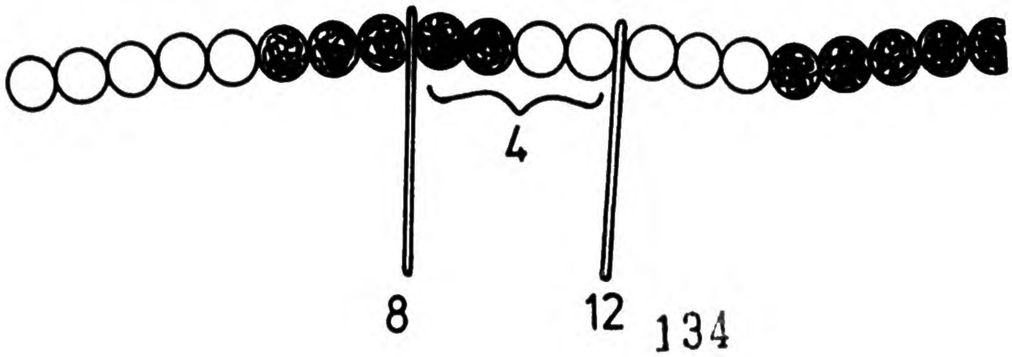
With money, two nickels make a dime. Five nickels (2½ dimes) make a quarter, and four quarters make a dollar. (So 'quarter' and 'fourth' have a common meaning.) Putting down a nickel and a penny, and below them a nickel and two pennies, shows a dime and three pennies (in value); $6 + 7 = 13$. Similarly $7 + 7 = 14$, etc. Also, $8 + 9$, in pennies, is two dimes minus three pennies, 17 (cents).

In schools where colored beads are a natural part of the equipment (less expensive than a computer!), after making necklaces and so on, *bead strings*, with color changing after each five beads (or after each ten for longer ones) give the best and quickest way of learning all sorts of number relations. Best is for two children to play together; learning is very rapid, and they expand the difficulty as they progress. (And the teacher is freed; the games spread.) The first games can be 'Show six' (put a toothpick down after the first bead of the second group), and "What number is this?" (asked after putting down a toothpick).

The diagram illustrates the result of "Show eight; now show four more; how many is that?" Or equally, show twelve, and now four less. *Doing* and *seeing* is understanding and remembering.

Remember that question of 26 children, four per car, how many cars, which 3% of the 9-year olds with calculators did? Ask 7-year olds who have used bead strings as a tool. Each bead is a child; put a toothpick after 26. Lay out fours with toothpicks, and the answer is there in about fifteen seconds. (Later it is *thought* through, still faster.)

Lay out sixes with toothpicks; a simple pattern appears (like a nickel-plus-penny repeated but with the nickels first). With color changes after each five, multiples of seven have a pattern like that of twos; *studying, playing with* these patterns gives the "times tables" beautiful meaning,



soon to be good friends, in place of the long dull meaningless drill with great failure that is so common.

What is 42, take away 18? You can *think* of a bead string (i.e., the decimal system) and (taking away the *first* eighteen) think: 18 to 20 to 40 to 42: 2 and 20 and 2, 24; or, eighteen to thirty-eight to thirty-twelve, twenty-four. (Did you *play* with such "funny language" with children, or say that "thirty-twelve" was *wrong*?)

Children working with *meanings* will not fall into confusion between 4×0, 4×1, $4 + 0$, a standard trap for most school children.

With large numbers, "But they don't understand place value!" is a perennial cry of the teachers. It is not that: the children simply are not using meanings, and may even get anxiety if asked to. "*Emphasize estimation*!" is the demand of the experts. But they are expert in technology, not in school; teaching this fails miserably, again since no meanings are used by the students.

The true solution is to *become familiar* with numbers of all sizes. And this can be done through discussions, in school and out, of all kinds of costs and prices; then writing and reading them. Here, one must give amounts of money in pennies, dimes, $ 1's, $ 10's, etc. "Cost of a car *and* a bicycle? But the insurance on the car is already far more than the cost of a new bike!" Estimation is coming in by itself. Separating the places in writing numbers by vertical lines (with all four operations) makes the working of the decimal system very clear (see an example below).

CONQUERING THE PROBLEM

It is the common belief, and strong belief of many or most research professionals, that many children cannot learn subtraction well and few can learn long division or fractions. But as preschoolers they learned far more complex things, without being taught. The trouble is simply that we have been trying to *teach* them instead of challenging them to find out. The actual learning is quite easy; the *crucial step* is to get them to drop their *attitude* of just trying to learn the right rules.

So Johnny (or a small group of people of any age) is with you, and you pick on the question 42 – 18, putting it in terms of money. "Here is some money you have" (leaving out preliminaries here); "these two plains (popsicle sticks) are dollars, and these (four) reds are tens. How much money do you have?" "Forty-two dollars". If there is trouble here, more play for familiarity with money is in order.

"Here is a nice picture, for eighteen dollars. Would you like to buy it?" Assent. "Can you say how much money you will have left?" This is to get him to think some; an answer is not needed.

"Well, if you want to buy the picture, you must pay for it". He hands you two reds. The numbers involved are understood. Now comes the challenge. "I am sorry, I have no change in this store".

The essential point is to leave it to Johnny. Instead of telling all, especially the "difficult" points, you tell nothing. *He* must learn to take responsibility. If there is an impasse, you could say "Let this be real. What might you then do?" When he has eyed the bag of money, show subtly that you let him choose what to do. A red is exchanged for a bundle of ten plains, and the picture can be bought and the change counted. (A teacher may at first feel impelled to give hints or explanations at the critical point. This is a half-way procedure that can develop soon enough into the full challenge which is much the best for long-term results.)

Next, ask him to repeat the experience, seeing just what is happening, and recording. When the recording, perhaps in improved form, shows the tens and ones, present at each stage, the algorithm

	$ 10	$ 1
have	4	2
get change	3	12
pay	1	8
have left	2	4

appears. Now let him do another problem, recording in full himself. The next step is to have, say, $ 504 and pay $ 87. There will be no difficulty in changing a $ 100 to ten $ 10's; then one of these tens must be exchanged for ten ones. The rest is now easy.

In school, whether for remedial work or right at the start, different groups may do different problems, and record. Other groups then read the recording, and reconstruct that experience. Thus *using* mathematical reasoning comes first, *seeing* the results and thus understanding comes next, then *writing* results, and finally *reading*. This natural order corresponds to true *learning through meanings*. Once the process is carried out in full and practiced some, it is *known*.

A final step is through *self-tests*. "Do you *know* that you can do these problems, and *be sure your results are correct*?" An answer must *come first*. "Then pick a hard one, and take what time you need; then *show*

me or someone exactly what you did". When this is done successfully, the student can *take responsibility* for that type of question, and one can *go on to other topics*. The pressure to *cover the material* can quickly disappear. Moreover the students, with *knowledge* that *they can think and do*, will find standardized tests (if they can be understood) easy. And this applies to *all* students, not just the 'gifted'.

The same basic method, *giving the students the responsibility*, works generally. For long division, let a group of say six people share some money, say \$ 2183 (with popsicle sticks, blue = \$ 100, yellow = \$ 1000). A 'Treasurer' may give each person a red at first, but soon they must find a way to share the thousands; so one of these will be exchanged for ten hundreds. The rest is generally quickly accomplished. Next, ask them to structure the work, using at each step all the biggest money first. On recording, they see the algorithm appear. The effect can be electric: "We used no math rules!" Reading comes next, as in the last example. The same method applied to decimals (to start, use dollars and cents).

For fractions, finding what various simple fractions of a dollar are, and putting them in order, is good; getting familiarity is the keynote. For instance, since $\frac{1}{3}$ of \$ 1 is $33\frac{1}{3}$ c, $\frac{2}{3}$ of \$ 1 is $66\frac{2}{3}$c, and half a quarter is $12\frac{1}{2}$c, so $\frac{5}{8}$ of \$ 1 is $62\frac{1}{2}$c, less than $\frac{2}{3}$ of \$ 1. Or, three times $\frac{2}{3}$ is 2 and 3 times $\frac{5}{8}$ is $\frac{15}{8}$ which is less than 2, giving the same result.

In the NAEP [6], all three age groups were asked to put the fractions $\frac{5}{8}$, $\frac{3}{10}$, $\frac{3}{5}$, $\frac{1}{4}$, $\frac{2}{3}$, $\frac{1}{2}$ in order. The success rates were

age 9: 0%; age 13: 2%; age 17: 12%.

If these students had had freedom to explore and get a good number sense, the results would have been radically different.

ISSUES

What I have been expressing above is in great conflict with certain beliefs coming especially from the research community. I find it essential to compare all these views and work to resolve the conflicts; for I see the recommendations I so deplore as coming naturally from those other views.

I quote especially the following:

(1) All learning comes from the teacher; all material is presented by the teacher.

(2) Only the gifted can learn school mathematics well and easily.

(3) Students (omitting the gifted) cannot solve problems involving two or more steps.

(4) Time on task is what counts. More specifically, "A firm generalization arising from research on learning of skills is that acquiring significant competence in domains of any difficulty requires large amounts of guided practice – much more than is provided for most pupils in schools or most adults who try to prepare for new technical jobs" [11].

(5) Slow learners, the disadvantaged in particular, cannot be expected ot get control over school mathematics.

No. 1, I find expressed continually in various articles. It seems to be an *attitude*: The teachers are there for the job of teaching and they must carry on the job; and teaching means presenting the material. (I regret that most mathematicians also fall partly into this trap; I did in my younger days.)

About the quoted beliefs, let me first say that I believe they represent fairly well *what we see in the school situation* normally and what we see in standardized tests. But 'can' and 'cannot' should be replaced by 'do' and 'do not', with the phrases added 'in the school situation'. Only in this way do the statements become objective and true. The reseachers seem ot hide from counterexamples. For instance, in the SEED program under W. F. Johntz, hundreds of mathematicians and users of mathematics have been leading regular daily classes, typically of fourth of fifth grade disadvantaged children. The subject is usually abstract algebra. Through questioning, the children are led into real mathematical discoveries with competence and joy. The work has been funded by several states and the USOE for years.

In 'time on task' the principal message seems to be that if students are showing essentially no learning day by day, it just requires years of work rather than days. The continued failure is clear, the learning is not. Just as likely, that later catching onto concepts comes from some moments of free time to ponder various relationships in one's own way, allowing them at last to emerge.

The divergence with what I am presenting is extreme; I see the continual teaching, i.e. presenting material, as a major cause of the disaster for the students. Thus, in fact, the researchers and I are examining in large part totally different worlds; one where the students are controlled and taught, and the other where, with less pressure and more responsibility, they grow in their own powers. We will be on our way to

beautiful results when the researchers enter the latter world (compare the last two sections) and find these powers in the students. This will, of course, require radically new attitudes; letting the student explore and search, without pressure (and increasingly without hints). This may at first seem careless and abandoning (to many researchers and teachers), but becomes supportive and relaxing.

Most striking and illuminating is the work quoted by Benezet [10], which explodes all the beliefs at once; and his methods involved essentially no teacher training, just some general instructions. His report is very interesting and illuminating, and should be made available to the general reader.

The greatest harm in the beliefs, to my mind, is that they are picked up and believed in the highest circles, thus, for instance, allowing the national commission recommendations about requiring more mathematics and work. This is my reason for stressing the need to examine the beliefs deeply. They also lead to the lowest of goals, passing standardized tests, which is associated with remembering piles of rules, a far cry from gaining control with responsibility over domains of thought, the essential for true progress in science, technology and elsewhere.

They also lead to the call for starting teaching earlier, with the great dangers I have spoken of. Benezet started simply by cutting out all formal arithmetic from the first two grades; the children caught up rapidly in grade 3, and the improvement was very clear (the children were then not held back in these grades, for instance).

WHAT CAN BE DONE?

The most pressing need I see is for us to face fully the consequences of interventions we make, and hold up on those with bad results. I speak, of course, of mandating more work in mathematics for failing students, raising standards for these without helping them toward meeting the standards, and starting mathematics teaching at an earlier age. It is unthinkable to market drugs without a thorough study of all effects; in education I see no concern.

All the states have published goals for education, but these goals seem ot be mostly disregarded in practice. Urgently needed is a clear goal for mathematics education, with clear school methods for working toward the goal. I suggest the following simple statements.

Goal for the future of mathematics students: Be able to study deeply into situations with mathematical elements, drawing and organizing con-

clusions, and to carry on oral and written communication about them with clarity; also to take full responsibility as to their correctness and relevance, and to look into various possible relationships with outside matters.

From this, a *school goal* is clear: Have plenty of practice in exploring such situations, with initiative and concentration, keeping all purposes in mind; and carrying on thoughtful discussions with growing responsibility.

Most teachers cannot be expected to institute such methods at once. But with support from others, letting the students start some explorations and seeing and discussing the needed changes in attitude, both teachers and students can sense the new growth and gradually get more involved; this will already ease up on the pressures to accomplish. Breaking the class into groups and individuals will naturally being, and motivation will become intrinsic. With some natural choices of situations, the standard curriculum will appear from being *used* (discovered) and hence be well understood; the usual tests become easy to pass. And the proof will be there that all students have high capability and can learn relatively quickly.

The presently designated 'effective schools' may show better climate for learning, which is of great importance; but with the present teaching methods, whether this learning is usable in practical situations is open to question (see the NAEP results).

The great problem is that the needed goals do not have priority, and the possibility of progress as described is not faced. There is a great need to get a national focus onto the basic questions of teaching and learning and how schooling can be changed to fit. For this, some central body is urgently needed to keep the top priority, coordinate and publicize. It could naturally be located in the National Science Foundation, especially for the scientific aspects, and would coordinate with professional societies. The actual studies can be carried out by present and newly formed groups, and in part be extensions of present work. I recommend making an informal start in this direction as soon as possible.

I end with a few remarks on particular topics.

Remedial work is now very common, from early grades into college. Unfortunately, it is mostly more of the same, leaving the rule-following attitudes untouched. Just as in original learning, it is these attitudes that must change for effective work; then one can start with simple ideas and work rapidly up. Thus a real cure is no different from original learning; see the section "Conquering the problem".

The *diagnose-prescribe* movement expresses a proper way to go about remediation; but so far I have not seen much beyond working at symptoms instead of the disease underneath, with correspondingly poor results likely.

Individualized instruction and *mastery learning* are fine sounding terms, but used mostly in programs governed by pre- and posttests. When run mechanically in large part, the student tends to get lost from view; he and she can get into a morass of misconceptions without its being discovered. Compare Erlwanger [12]. Like any program based on testing, the goals are most likely to be far removed from what is truly needed. And I see 80% correct as 20% wrong, thus without the solidity needed for real life.

Of course I can only give some general trends I have seen; any individual case must be judged on its own merits. Finally, a great long-term need is for mathematics and science professionals to work with local schools, helping the understanding of true long-term goals, helping teachers become comfortable with these goals and move the teaching methods in that direction. They can help the whole community recognize the importance of changed attitudes and methods and hence support improvement for all the students as the work gets underway.

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